About The Lascar Group

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DISSERTATION

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To Enrique, Xavier and Alf

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Contents

Introduction

This thesis is intended to provide the reader with new results about the Lascar group and the notion of G-compactness. Specifically we deal with the following three issues:

- *The definition of the Lascar group without using saturated models.* Instead of using saturated models whose existence depends on extra set theoretical assumptions, we work with $|T|^+$-resplendent models, which always exist.
- *The non preservation of G-compactness under adding parameters to the language.* It was an open question wether G-compactness was a robust property in this sense. We answer this question negatively offering examples of a theory T and a set of parameters A where T is G-compact over $\emptyset$ but T_A is not.
- *The existence of a one-sorted ω-categorical non G-compact structure.* Ivanov constructed in [10] a structure like this, and here we prove a more general theorem from which we can easily derive the result.

In [13] Lascar introduced the group $\mathrm{Autf}(N/A)$ of strong automorphisms of N over A. The quotient $\mathrm{Aut}(N/A)/\mathrm{Autf}(N/A)$ is independent of the choice of N (for a big saturated model N and a small subset $A \subseteq N$) and it is now called *the Lascar group over* A. Lascar showed in [13] that in the case of a very large class of theories, called by him *G-compact*, the group carries a compact Hausdorff topology. In the last decade the Lascar group has received a lot of attention, particularly because of its importance for simple theories and hyperimaginaries, and also because of the discovery of non G-compact theories. It is now known that the Lascar group is a compact (not necessarily Hausdorff) topological group for any first-order theory. In the case of the theory of an algebraically closed field it corresponds to the absolute Galois group over the field generated by A. In the first chapter we present some well known results about the Lascar group and we refer to [14], [4] and [25] for more details.

In the second chapter we present the material from [5]. Working with the class of $|T|^+$-resplendent models we obtain two main results: One indicates that $\mathrm{Autf}(N/A)$ can be characterized (similarly to what Lascar originally did working with saturated models) as the least *very normal* subgroup of $\mathrm{Aut}(N/A)$, we call it $\Gamma(N/A)$, which is its least normal subgroup closed under a more general conjugation that we call *weak conjugation.*

(Theorem 2.3.11) *For any $|T|^+$-resplendent model N and any $A \subseteq N$ such that $|A| \leq |T|$, $\mathrm{Autf}(N/A) = \Gamma(N/A)$.*

The other one shows that we can define the Lascar group using $|T|^+$-resplendent models.

(Theorem 2.3.14) *For any $|T|^+$-resplendent model N and any $A \subseteq N$ such that $|A| \leq |T|$, $\mathrm{Aut}(N/A)/\mathrm{Autf}(N/A)$ is independent of the choice of N.*

The proofs are quite different than the ones by Lascar in [13]. We strongly use the properties of resplendency and avoid completely the use of ultraproducts. At the end of the chapter we show that these results also hold in the wider class of all $|T|^+$-saturated and strongly $|T|^+$-homogeneous models.

In the third chapter we deal with the notion of G-compactness. In section 3.1 we present the first examples of non G-compact theories obtained in [4], and we make use of them to build, in section 3.2, three examples of a theory T and a set of parameters A which illustrate the fact that G-compactness is not preserved under adding constants to the language. Having in mind that

([14], Remark 4.20) *The following are equivalent:*

1. *T is G-compact (over A).*
2. *$\overset{\mathrm{L}}{\equiv}_A = \overset{\mathrm{KP}}{\equiv}_A$ (even for infinite tuples).*
3. *$\overset{\mathrm{L}}{\equiv}_A = \overset{\mathrm{KP}}{\equiv}_A$ for finite tuples and $\mathrm{Autf}(\mathfrak{C}/A)$ is closed in $\mathrm{Aut}(\mathfrak{C}/A)$ (with the point-wise topology),*

the set of counterexamples is complete in the sense that the equality $\overset{\mathrm{L}}{\equiv}_A = \overset{\mathrm{KP}}{\equiv}_A$ for finite tuples is preserved from T to T_A in the first to examples, but not in the third one. Section 3.3 is dedicated to give a new proof of a result originally proved by Newelski in [16]:

(Corollary 3.3.7) *([16], Corollary 1.8) Type-definable Lascar strong types have finite diameter.*

In the proof we make use of the notions of *c-free* and *weakly c-free* extensions introduced also by Newelski in [17].

In the last chapter we tackle the problem of the existence of a one-sorted ω-categorical non G-compact structure. We derive its existence from one of the examples of non G-compact theories presented in [4] and the following general theorem that we prove in section 4.2.

(Theorem 4.1.1) *Let T' be a many-sorted ω-categorical theory with countably many sorts. Then there is a one-sorted ω-categorical theory T^* in which T' is stably embeddable.*

For the proof of this theorem we make use of a theory that we call T_E in the language of infinitely many equivalence relations. This theory is interesting on its own from the Shelah's classification point of view. In particular we prove that T_E is not simple (**Theorem 4.3.4**) and does not have SOP_1 (**Theorem 4.3.5**).

Finally, let us set some conventions on notation and terminology. In general, T will denote an arbitrary complete first-order theory with infinite models. Usually its language will be $\mathcal{L}$ and $\mathfrak{C}$ will be its monster model, which we think of as a model whose universe is a proper class and which realizes any type over any subset. The existence of the monster model can be guaranteed in any theory and does not require any additional hypothesis. All models we consider will be elementary submodels of $\mathfrak{C}$. If M is a model and $A \subseteq M$ is a set of parameters, $\mathcal{L}(A)$ is the expanded language with names for all elements of A; M_A is the standard expansion of M to $\mathcal{L}(A)$ where every element of A has its corresponding name, and $T_A = \mathrm{Th}(M_A)$ is its first-order theory. An A-automorphism of a model $M \supseteq A$ is an automorphism f of M which is the identity on A. It is also called an automorphism of M over A. The group of all A-automorphisms of M is denoted $\mathrm{Aut}(M/A)$. When we speak of the type $\mathrm{tp}(M/A)$ of a model M over a set A, we implicitly assume an enumeration of the model M. By $\mathrm{qftp}(a/A)$ we denote the quantifier free type of a over a A, and sometimes we write it with a subindex $\mathrm{qftp}_M(a/A)$, or even $\mathrm{qftp}_{\mathcal{L}}(a/A)$, to make clear (if necessary) what is the model or the language in which the type is considered.

1

Background

1.1 The Lascar group and G-compactness.

In this section we recall some well known facts about the Lascar Group, Lascar strong types and G-compactness. Let T be a complete first order theory in a language $\mathcal{L}$ and let $\mathfrak{C}$ be its monster model. In [13], Lascar introduced the following groups for a given set of parameters $A \subseteq \mathfrak{C}$:

1. $\mathrm{Autf}(\mathfrak{C}/A)$, the *Lascar strong automorphisms* (over A), which is the (normal) subgroup of $\mathrm{Aut}(\mathfrak{C}/A)$ generated by $\bigcup_{A \subseteq M \prec \mathfrak{C}} \mathrm{Aut}(\mathfrak{C}/M)$.
2. $\mathrm{Gal_L}(T_A)$, the *Lascar (Galois) group of* T (over A), obtained as the quotient group $\mathrm{Aut}(\mathfrak{C}/A)/\mathrm{Autf}(\mathfrak{C}/A)$.

As we will see in the next chapter, it is not necessary to work within the monster model; it is enough to assume that $\mathfrak{C}$ is $|T|^+$-saturated and strongly $|T|^+$-homogeneous. Lascar showed in [13] that in the case of a very large class of theories, called by him *G-compact*, $\mathrm{Gal_L}(T_A)$ carries a compact Hausdorff topology. Later he found that in general, as Ziegler shows in [25], $\mathrm{Gal_L}(T_A)$ is a quasicompact (compact but not necessarily Hausdorff) topological group for any first-order theory, even in a non G-compact one. We describe briefly the topology presented there (to simplify notation, let $A = \emptyset$).

Lemma 1.1.1. *([25], Lemma 1) Let M, N enumerate two small (of cardinality $|T|$) submodels $\mathfrak{C}$ and let $f \in \mathrm{Autf}(\mathfrak{C})$. The class of f in $\mathrm{Gal_L}(T)$ is determined by the type of $f(M)$ over N.*

If we fix two enumerations M, N of two small submodels of $\mathfrak{C}$, we can define a surjective map μ from $\mathrm{Aut}(\mathfrak{C})$ to $S_M(N) = \{\mathrm{tp}(M'/N) : M' \equiv M\}$ sending every $f \in \mathrm{Aut}(\mathfrak{C})$ to the type $\mathrm{tp}(f(M)/N)$. By the previous result, the quotient map from $\mathrm{Aut}(\mathfrak{C})$ to $\mathrm{Gal_L}(T)$ factors through μ:

$$\mathrm{Aut}(\mathfrak{C}) \xrightarrow{\mu} S_M(N) \xrightarrow{\nu} \mathrm{Gal}_{\mathrm{L}}(T)$$

Since $S_M(N)$ is a closed subspace of $S_{|T|}(N)$, it is a boolean space. $\mathrm{Gal}_{\mathrm{L}}(\mathrm{T})$ is endowed with the quotient topology with respect to μ, i.e., a subset $C \subseteq \mathrm{Gal}_{\mathrm{L}}(T)$ is closed iff $\nu^{-1}(C)$ is closed in $S_M(N)$. This definition does not depend on the choice of M and N, and with this topology $\mathrm{Gal}_{\mathrm{L}}(T)$ becomes a quasicompact group ([25], Lemma 10). In the case $\mathrm{Gal}_{\mathrm{L}}(T_A)$ is Hausdorff we say that that the theory T is *G-compact* (over A). Lascar showed in [13] that stable theories are G-compact and asked if there were any non G-compact theories. Some years later, Kim and Pillay ([11],[12]) showed that simple theories were also G-compact, and the first examples of non G-compact theories were given in [4].

We say that two tuples (possibly infinite) of the same length $a, b \in \mathfrak{C}$ have the same *Lascar strong type over A*, and we denote it by $a \overset{\mathrm{L}}{\equiv}_A b$, if they lie in the same orbit under $\mathrm{Autf}(\mathfrak{C}/A)$. The relation $\overset{\mathrm{L}}{\equiv}_A$ on tuples of a fixed length κ ($\kappa \leq |T|$) is clearly A-invariant and bounded (boundedly many classes) since, by the previous lemma, $\left|\mathfrak{C}^\kappa / \overset{\mathrm{L}}{\equiv}_A\right| \leq \left|S_{|T|+|A|}(N)\right| \leq 2^{|T|+|A|}$. Moreover, $\overset{\mathrm{L}}{\equiv}_A$ is precisely the finest bounded A-invariant equivalence relation on tuples of $\mathfrak{C}$ of a fixed length ([12], proposition 5.4).

There is also the notion of the finest bounded type-definable (over A) equivalence relation. Given two tuples (possibly infinite) of the same length $a, b \in \mathfrak{C}$, we say that they have the same *Kim-Pillay type over A*, and we denote it by $a \overset{\mathrm{KP}}{\equiv}_A b$, if a, b are related under every type-definable (over A) bounded equivalence relation on tuples of the appropriate length. The next result is known from [11], [12] and [14].

Fact 1.1.2. *([14], Remark 4.20) The following are equivalent:*

1. *T is G-compact (over A).*
2. *$\overset{\mathrm{L}}{\equiv}_A = \overset{\mathrm{KP}}{\equiv}_A$ (even for infinite tuples).*
3. *$\overset{\mathrm{L}}{\equiv}_A = \overset{\mathrm{KP}}{\equiv}_A$ for finite tuples and* $\mathrm{Autf}(\mathfrak{C}/A)$ *is closed in* $\mathrm{Aut}(\mathfrak{C}/A)$ *(with the pointwise topology).*

There is also another useful characterization of G-compactness. For a given set of parameters $A \subseteq \mathfrak{C}$, define a distance function d_A on tuples of $\mathfrak{C}$ of a fixed length (possibly infinite) by letting $d_A(a, b)$ ($a \neq b$) be the minimal natural number n (if it exists) such that for some $a_0 = a, a_1, \ldots, a_n = b$ tuples of $\mathfrak{C}$ of the same length, a_i, a_{i+1} can be extended to an infinite A-indiscernible sequence for every $i < n$. If no such n exists, then set $d_A(a, b) = \infty$, and if $a = b$, set $d(a, b) = 0$.

Lemma 1.1.3. *([25], Lemma 6 and 7) Let $A \subseteq \mathfrak{C}$, and $a, b \in \mathfrak{C}$ tuples of the same length.*

1. *If a, b can be extended to an infinite A-indiscernible sequence, then there is a model M ($A \subseteq M$) such that $a \equiv_M b$.*

2. *If $a \equiv_M b$ for some model $A \subseteq M$, then there is $c \in \mathfrak{C}$ (of the same length) such that a, c and b, c can both be extended to infinite A-indiscernible sequences.*

From this result we know that if the tuples a, b have the same type over a model containing A, then $d_A(a, b) \leq 2$. Moreover, we can easily see that for tuples a, b of $\mathfrak{C}$, $a \overset{\mathrm{L}}{\equiv}_A b \Leftrightarrow d_A(a, b) < \infty$.

Assume that a is a tuple of $\mathfrak{C}$ and consider $X = \left\{a' \in \mathfrak{C} : a' \overset{\mathrm{L}}{\equiv}_A a\right\}$, the Lascar strong type of a over A. Let $\operatorname{diam}_A(X)$, the A-diameter of X, be the supremmum of $d_A(a, b)$, for $b \in X$. In [16], Newelski proves the following criterion for G-compactness.

Theorem 1.1.4. *([16], Corollary 1.9) T is G-compact over A iff there is a finite bound for the A-diameters of Lascar strong types (of any length) over A.*

Regarding unstable theories, from the following fact we know that o-minimal theories are also G-compact.

Fact 1.1.5. *([25], Lemma 24) Every automorphism of a big saturated o-minimal structure is Lascar strong.*

The Lascar group has received a lot of attention in the last decade, particularly because of its importance for simple theories and hyperimaginaries and because of the discovery of non G-compact theories presented for the first time in [4].

1.2 Thick formulas

Let x, y be finite tuples of variables of the same length. We say that a formula $\varphi(x, y)$ is *thick* if it is symmetric and there is a finite k for which there is no sequence $(a_i : i < k)$ such that $\neg\varphi(a_i, a_j)$ for all $i < j < k$. For any set A and sequences of variables (possibly infinite) of the same length, the set of all thick formulas over A in (finite subtuples of) the variables x, y, is denoted by $\mathrm{nc_A}(x, y)$, and for any natural number $n > 1$, $\mathrm{nc_A}^n(x, y)$ denotes the type

$$\exists y_1, \dots y_{n-1}(\mathrm{nc_A}(x, y_1) \wedge \mathrm{nc_A}(y_1, y_2) \wedge \dots \wedge \mathrm{nc_A}(y_{n-1}, y))$$

If $\varphi(x, y)$ and $\theta(x, y)$ are thick formulas, then $\varphi(x, y) \vee \theta(x, y)$ is thick and using Ramsey's theorem one can see that $\varphi(x, y) \wedge \theta(x, y)$ is also thick.

It's worth noting that any type-definable bounded equivalence relation E (on tuples of possibly infinite length) can be defined by

$$E(x, y) \Leftrightarrow \bigwedge_{i \in I} E_i(x, y)$$

for some index set I, where each $E_i(x, y)$ is again a type-definable bounded equivalence relation which can be defined by

$$E_i(x, y) \Leftrightarrow \bigwedge_{n<\omega} \theta_n^i(x, y),$$

where each $\theta_n^i(x, y)$ is a thick formula. By compactness we may assume, moreover, that for all $n < \omega$,

$$\theta_{n+1}^i(x, z) \wedge \theta_{n+1}^i(z, y) \vdash \theta_n^i(x, y).$$

Lemma 1.2.1. *([25], Lemma 6) For any $a, b \in \mathfrak{C}$, $\models \mathrm{nc_A}(a, b)$ if and only if a, b can be extended to an infinite A-indiscernible sequence.*

By the previous lemma and lemma 1.1.3, we see that $\overset{\mathrm{L}}{\equiv}_A$ is just the transitive closure of the type-definable relation $\mathrm{nc_A}(x, y)$, and therefore it is defined by the infinite disjunction $\bigvee_{n<\omega} \mathrm{nc_A}{}^n(x, y)$. Making use of the Independence Theorem over a model for simple theories ([12],Theorem 3.5), one can show the following fact.

Fact 1.2.2. *([11], Proposition 13) Let T be simple. Then $a \overset{\mathrm{L}}{\equiv}_A b$ if and only if $d_A(a, b) \leq 2$.*

This shows, in particular, that simple theories are G-compact and $\overset{\mathrm{L}}{\equiv}_A$ is type-definable over A by the type $\exists z(\mathrm{nc_A}(x, z) \wedge \mathrm{nc_A}(z, y))$.

1.3 Imaginaries, hyperimaginaries, and Galois correspondence

As usual, let T be a complete first order theory and $\mathfrak{C}$ its monster model. For any set $A \subset \mathfrak{C}$, an *A-imaginary* is the class $[a]_E$ of a finite tuple $a \in \mathfrak{C}^n$ under an A-definable equivalence relation E on $\mathfrak{C}^n$ for some $n \in \omega$. An *imaginary* is just a $\emptyset$-imaginary. Shelah introduced imaginaries and the imaginary universe $\mathfrak{C}^{\mathrm{eq}}$ in [19] to prove the existence of canonical basis for stationary types in stable theories. In [15], Makkai proposed to construct $\mathfrak{C}^{\mathrm{eq}}$ as a many-sorted structure and this became the usual way to represent it. For every $\emptyset$-definable equivalence relation $E(x_1, \ldots, x_n, y_1, \ldots, y_n)$ on $\mathfrak{C}^n$, he added a new sort $\mathfrak{C}^n/E$ and a new function symbol π_E to the language $\mathcal{L}$ for the projection $\mathfrak{C} \to \mathfrak{C}^n/E$. The elements of $\mathfrak{C}^{\mathrm{eq}}$ are precisely the *imaginary* elements. T^{eq} denotes the complete theory of $\mathfrak{C}^{\mathrm{eq}}$ in the new language $\mathcal{L}^{\mathrm{eq}} = \mathcal{L} \cup \{\pi_E : E \text{ is a } \emptyset\text{-definable equivalence relation}\}$ and $\mathfrak{C}^{\mathrm{eq}}$ turns out to be the monster model of T^{eq}.

Lemma 1.3.1. *([24], Lemma 1.2) For every $\mathcal{L}^{\mathrm{eq}}$-formula $\varphi(\overline{y}; x_1^{E_1}, \ldots, x_n^{E_n})$, where $\overline{y}$ is a tuple of variables of the real sort and, for each i, the variable $x_i^{E_i}$ belongs to* *the sort $\mathfrak{C}^{n_i}/E_i$, there is an $\mathcal{L}$-formula $\psi(\overline{y};\overline{y}_1,\ldots,\overline{y}_n)$ such that for arbitrary tuples $\overline{a},\overline{a}_1,\ldots,\overline{a}_n \in \mathfrak{C}$ of the appropriate length,*

$$\mathfrak{C}^{\mathrm{eq}} \models \varphi(\overline{a};\pi_{E_1}(\overline{a}_1),\ldots,\pi_{E_n}(\overline{a}_n)) \Leftrightarrow \mathfrak{C} \models \psi(\overline{a};\overline{a}_1,\ldots,\overline{a}_n)$$

From this fundamental result we can see that any automorphism $f \in \mathrm{Aut}(\mathfrak{C})$ extends uniquely to an automorphism $f' \in \mathrm{Aut}(\mathfrak{C}^{\mathrm{eq}})$. For any $A \subseteq \mathfrak{C}$, $\mathrm{dcl}^{\mathrm{eq}}(A)$ denotes the *imaginary definable closure of* A, i.e., the set

$$\{e \in \mathfrak{C}^{\mathrm{eq}} : |\{f(e) : f \in \mathrm{Aut}(\mathfrak{C}^{\mathrm{eq}}/A)\}| = 1\}$$

of elements that are fixed by $\mathrm{Aut}(\mathfrak{C}^{\mathrm{eq}}/A)$, and $\mathrm{acl}^{\mathrm{eq}}(A)$ denotes the *imaginary algebraic closure of* A, i.e., the set

$$\{e \in \mathfrak{C}^{\mathrm{eq}} : |\{f(e) : f \in \mathrm{Aut}(\mathfrak{C}^{\mathrm{eq}}/A)\}| < \omega\}$$

of elements that have a finite orbit under $\mathrm{Aut}(\mathfrak{C}^{\mathrm{eq}}/A)$. We say that two tuples (possibly infinite) of the same length $a, b \in \mathfrak{C}$ have the same *strong type over* A ($a \overset{\mathrm{stp}}{\equiv}_A b$) if $\mathrm{tp}(a/\mathrm{acl}^{\mathrm{eq}}(A)) = \mathrm{tp}(b/\mathrm{acl}^{\mathrm{eq}}(A))$. Moreover, for finite tuples, $a \overset{\mathrm{stp}}{\equiv}_A b$ if and only if $\models E(a,b)$ for every A-definable finite equivalence relation E with finitely many classes. The *strong type of* a *over* A is just the type $\mathrm{tp}(a/\mathrm{acl}^{\mathrm{eq}}(A))$ and it is denoted by $\mathrm{stp}(a/A)$.

Similarly, for any set $A \subseteq \mathfrak{C}$, an A-*hyperimaginary* is an equivalence class $[a]_E$ (for simplicity in notation just a_E) of a sequence (possibly infinite) a of elements of $\mathfrak{C}$ under a type-definable over A equivalence relation E. Clearly A-imaginaries are A-hyperimaginaries and a *hyperimaginary* is just a $\emptyset$-hyperimaginary. $\mathfrak{C}^{\mathrm{heq}}$ denotes the class of hyperimaginaries. If a is of the form $a = (a_i : i < \alpha)$ for some ordinal α, we say that α is the *length* of the hyperimaginary a_E. *Finitary hyperimaginaries* are hyperimaginaries of finite length. We say that an automorphism $f \in \mathrm{Aut}(\mathfrak{C})$ fixes a hyperimaginary a_E if $f(a_E) = a_E$, that is, $\models E(a, f(a))$.

Let A be a class of hyperimaginaries. The *hyperimaginary definable closure of* A, denoted by $\mathrm{dcl}^{\mathrm{heq}}(A)$, is the class of all hyperimaginaries which are fixed by the automorphisms fixing pointwise A, that is,

$$\mathrm{dcl}^{\mathrm{heq}}(A) = \left\{e \in \mathfrak{C}^{\mathrm{heq}} : |\{f(e) : f \in \mathrm{Aut}(\mathfrak{C}/A)\}| = 1\right\}.$$

The *hyperimaginary algebraic closure of* A, denoted by $\mathrm{acl}^{\mathrm{heq}}(A)$, is the class of all hyperimaginaries having a finite orbit under the group of all automorphisms fixing pointwise A, that is,

$$\mathrm{acl}^{\mathrm{heq}}(A) = \left\{e \in \mathfrak{C}^{\mathrm{heq}} : |\{f(e) : f \in \mathrm{Aut}(\mathfrak{C}/A)\}| < \omega\right\}.$$

And the *bounded closure of* A, denoted by $\mathrm{bdd}(A)$, is the class of all hyperimaginaries having a bounded orbit under the group of all automorphisms fixing pointwise A, that is,

$$\mathrm{bdd}(A) = \left\{e \in \mathfrak{C}^{\mathrm{heq}} : |\{f(e) : f \in \mathrm{Aut}(\mathfrak{C}/A)\}| < |\mathfrak{C}|\right\}.$$

If a is a sequence of hyperimaginaries, $\mathrm{dcl}^{\mathrm{heq}}(a) = \mathrm{dcl}^{\mathrm{heq}}(A)$, where A is the set enumerated by a. We say that two sequences of hyperimaginaries a, b are equivalent if $\mathrm{dcl}^{\mathrm{heq}}(a) = \mathrm{dcl}^{\mathrm{heq}}(b)$. Using compactness one can easily check the following remark.

Remark 1.3.2. *For any class of imaginaries A,*

1. $\mathfrak{C}^{\mathrm{eq}} \cap \mathrm{dcl}^{\mathrm{heq}}(A) = \mathrm{dcl}^{\mathrm{eq}}(A)$.

2. $\mathfrak{C}^{\mathrm{eq}} \cap bdd(A) = \mathfrak{C}^{\mathrm{eq}} \cap \mathrm{acl}^{\mathrm{heq}}(A) = \mathrm{acl}^{\mathrm{eq}}(A)$.

The following result follows immediately from Lemma 1.7. in [14].

Lemma 1.3.3. *For any set of hyperimaginaries A, there are hyperimaginaries e, f such that* $\mathrm{bdd}(A) = \mathrm{dcl}^{\mathrm{heq}}(e)$ *and* $\mathrm{acl}^{\mathrm{heq}}(A) = \mathrm{dcl}^{\mathrm{heq}}(f)$.

Theorem 1.3.4. *([14], Theorem 4.15) Let $e \in \mathrm{bdd}(\emptyset)$. Then e is equivalent to some sequence of finitary bounded hyperimaginaries.*

Let a_E, b_F be two hyperimaginaries where E, F are $\emptyset$-type-definable equivalence relations on sequences of elements of $\mathfrak{C}$, and let

$$\Gamma = \{\varphi(x, y) \in L : \exists a' E a, \exists b' F b, \models \varphi(a', b')\}.$$

Define the type of a_E over b_F as the set of formulas

$$\mathrm{tp}(a_E/b_F) = \bigcup_{\varphi(x,y)\in\Gamma} \exists x' y' (E(x, x') \wedge F(b, y') \wedge \varphi(x', y')).$$

It is a partial type over b, let's say $\pi(x, b)$, and the following properties can be easily checked.

Remark 1.3.5. *Let a_E, b_F be two hyperimaginaries and* $\pi(x, b) = \mathrm{tp}(a_E/b_F)$.

1. *For any $b' F b$, $\pi(x, b) \equiv \pi(x, b')$.*

2. *For any $a' E a$, $\models \pi(a', b)$.*

3. *For any a', $\models \pi(a', b)$ iff there is $f \in \mathrm{Aut}(\mathfrak{C}/b_F)$ such that $f(a_E) = a'_E$.*

A *complete type over a hyperimaginary e* in the real variables x is a type of the form $p(x) = \mathrm{tp}(a/e)$, where $a \in \mathfrak{C}$ is a sequence of the length of x. It is a partial type but it is complete in the sense of the third point in the previous remark, i.e., for any $a, b \models p(x)$, there is some $f \in \mathrm{Aut}(\mathfrak{C}/e)$ such that $f(a) = b$.

Fact 1.3.6. *([6], Proposition 15.16) For any set $A \subseteq \mathfrak{C}$ and tuples (possibly infinite) $a, b \in \mathfrak{C}$, $a \overset{\text{KP}}{\equiv}_A b$ if and only if* $\text{tp}(a/\text{bdd}(A)) = \text{tp}(b/\text{bdd}(A))$.

Fact 1.3.7. *([14], Remark 4.8)* $\text{Gal}_{\text{L}}(T)$ *acts on the set of bounded hyperimaginaries.*

Now we can state the result which establishes a Galois correspondence between (definably closed) sets of hyperimaginaries and closed subgroups of $\text{Gal}_{\text{L}}(T)$.

Theorem 1.3.8. *([14], Corollary 4.16) There is a Galois correspondence between closed subgroups H of* $\text{Gal}_{\text{L}}(T)$ *and definably closed sets A of finitary bounded hyperimaginaries:*

- $H_A = \{g \in \text{Gal}_{\text{L}}(T) : g(a) = a$ *for all* $a \in A\}$.
- $A_H =$ *the set of all finitary bounded hyperimaginaries a such that $g(a) = a$ for all $g \in H$.*

1.4 Stable embeddability

In the last chapter we present some results that involve the notions of *stably embedded* sets and *stably embeddable* theories. In this section we present the basic material which will be needed.

Let T be a complete theory with monster model $\mathfrak{C}$, let p be a (partial) m-type over the empty set and P the set of of realizations of p in $\mathfrak{C}$, together with the structure induced from $\mathfrak{C}$, i.e., the 0-definable subsets of P^n are the traces on P^n of 0-definable subsets of $\mathfrak{C}^{m\cdot n}$. We say that P is *stably embedded* if for every $n \in \omega$, if $D \subseteq \mathfrak{C}^{m\cdot n}$ is definable, then $D \cap P^n$ is relatively definable with parameters from P. The following is another version, with its proof, of a result presented in the appendix of [7], and it can also be adapted to the case where P is a collection of sorts. We don't assume elimination of imaginaries and we allow the language to be arbitrarily large.

Lemma 1.4.1. *The following conditions are equivalent:*

(1) For every a, $\text{tp}(a/\text{dcl}^{\text{eq}}(a) \cap \text{dcl}^{\text{eq}}(P)) \vdash \text{tp}(a/P)$.

(2) For every a, there is a set $P_0 \subseteq P$, $|P_0| \leq |T| + |a|$, such that $\text{tp}(a/P_0) \vdash \text{tp}(a/P)$.

(3) For every a, there is a set $P_1 \subseteq P$, $|P_1| \leq |T| + |a|$, such that $\text{tp}(a/\text{acl}(P_1)) \vdash \text{tp}(a/P)$.

(4) For every a, $\text{tp}(a/P)$ *is definable over some $P_0 \subseteq P$, $|P_0| \leq |T| + |a|$.*

(5) P is stably embedded.

(6) Every automorphism of P lifts to an automorphism of $\mathfrak{C}$.

Proof. (1) → (2). For each $x \in \mathrm{dcl}^{\mathrm{eq}}(a) \cap \mathrm{dcl}^{\mathrm{eq}}(P) = B$, let B_x be a finite subset of P, such that $x \in \mathrm{dcl}^{\mathrm{eq}}(B_x)$. Let $P_0 = \bigcup_{x \in B} B_x$. Observe that P_0 has the desired cardinality and $\mathrm{dcl}^{\mathrm{eq}}(a) \cap \mathrm{dcl}^{\mathrm{eq}}(P) \subseteq \mathrm{dcl}^{\mathrm{eq}}(P_0)$, thus, by assumption,

$$\mathrm{tp}(a/P_0) \vdash \mathrm{tp}(a/\mathrm{dcl}^{\mathrm{eq}}(P_0)) \vdash \mathrm{tp}(a/\mathrm{dcl}^{\mathrm{eq}}(a) \cap \mathrm{dcl}^{\mathrm{eq}}(P)) \vdash \mathrm{tp}(a/P).$$

(3) → (4). Let $\varphi(x, y)$ be a formula, and let P_1 be given by (3). We will first see that $\mathrm{tp}(a/P) \restriction \varphi$ has a definition over $\mathrm{acl}(P_1)$. By assumption, we know that

$$p(y_1) \cup p(y_2) \cup \{\psi(y_1) \leftrightarrow \psi(y_2) : \psi(y) \in L(\mathrm{acl}(P_1))\} \vdash \varphi(a, y_1) \leftrightarrow \varphi(a, y_2)$$

and, by compactness, there are $\psi_1(y), \ldots, \psi_k(y) \in L(\mathrm{acl}(P_1))$ such that

$$p(y_1) \cup p(y_2) \cup \{\psi_i(y_1) \leftrightarrow \psi_i(y_2) : 1 \leq i \leq k\} \vdash \varphi(a, y_1) \leftrightarrow \varphi(a, y_2).$$

Now, for each finite sequence $s \in 2^k$, let $\theta_s = \bigwedge_{1 \leq i \leq k} \psi_i^{s(i)}$, where $\psi_i^1 = \psi_i$ and $\psi^0 = \neg\psi_i$. Observe that $P \cap \varphi(a, \mathfrak{C})$ is relatively defined by a disjunction, say $\psi(y) = \bigvee_{1 \leq i \leq l} \theta_{s_i}$, for some $s_1, \ldots, s_l \in 2^k$. $\psi(y)$ is the definition we wanted.
Let $B = \{b \in P : \models \psi(b)\}$. Since B is relatively definable over $\mathrm{acl}(P_1)$, it has finitely many conjugates over P_1, say $B_0, \ldots B_n$. Now consider the following equivalence relation

$$E(y_1, y_2) \Leftrightarrow \bigwedge_{i \leq n} y_1 \in B_i \leftrightarrow y_2 \in B_i$$

and choose a set $\{b_0, \ldots b_m\} \subseteq P$ of representatives of all the E-classes. Let $P_2 = P_1 \cup \{b_0, \ldots b_m\}$ and observe that for every $f \in \mathrm{Aut}(\mathfrak{C}/P_2)$, $f(B) = B$. Thus B is relatively definable over P_2 by a formula which we call $\psi'(y)$. Notice that

$$\begin{aligned} \varphi(x, b) \in \mathrm{tp}(a/P) \quad &\Leftrightarrow \quad \models \varphi(a, b) \\ &\Leftrightarrow \quad \models \psi(b) \\ &\Leftrightarrow \quad \models \psi'(b), \end{aligned}$$

thus $\mathrm{tp}(a/P) \restriction \varphi$ has a definition over P_2. If we do the same for every formula $\varphi(x, y)$, we obtain a set, say $P' \subset P$ of the desired cardinality, over which $\mathrm{tp}(a/P)$ is defined.
(4) → (5). Let $D \subset P^n$ be relatively defined by the formula $\varphi(x, b)$, and let $d_\varphi(y)$ be a definition for $\mathrm{tp}(a/P) \restriction \varphi$ over P. Then $d_\varphi(y)$ relatively defines the set $D \subseteq P^n$.
(5) → (4). Let a be a tuple of $\mathfrak{C}$ and $\varphi(x, y)$ a formula. We want to find a definition over P for $\mathrm{tp}(a/P) \restriction \varphi$. By assumption, we know that

$$\{b \in P : \models \varphi(a, b)\} = \{b \in P : \psi(b)\}$$

for some $\psi(y) \in L(P)$. $\psi(y)$ is the definition we wanted.

(2) $\rightarrow$ (6). Let A, B be subsets of $\mathfrak{C}$ and let $\tau : P \cup A \longrightarrow P \cup B$ be an elementary map such that $\tau(P) = P$. For each $a \in \mathfrak{C}$, we want to find $b \in \mathfrak{C}$ such that $\tau \cup (a, b)$ is still elementary, i.e., we want to find $b \in \mathfrak{C}$ satisfying $\mathrm{tp}^{\tau}(a/PA)$. By a back-and-forth argument we would finish. By assumption, there is a set P_0 such that $\mathrm{tp}(aA/P_0) \vdash \mathrm{tp}(aA/P)$, thus $\mathrm{tp}(a/AP_0) \vdash \mathrm{tp}(a/AP)$. Since τ is elementary, we have $\mathrm{tp}^{\tau}(a/AP_0) \vdash \mathrm{tp}^{\tau}(a/AP)$. Now it is enough to choose $b \in \mathfrak{C}$ a realization of $\mathrm{tp}^{\tau}(a/AP_0)$ by saturation of $\mathfrak{C}$.

(6) $\rightarrow$ (5). Assume P is not stably embedded. Let $S \subseteq P^n$ be relatively definable over $\mathfrak{C}$ but not over P. Then, for every subset $P_0 \subseteq P$, there is $f \in \mathrm{Aut}(\mathfrak{C}/P_0)$ such that $f(S) \neq S$.

First observe that S has infinitely many conjugates over $\emptyset$. Assume not, and let $\{S_0, \ldots, S_n\}$ be its orbit under $\mathrm{Aut}(\mathfrak{C})$. Consider the equivalence relation defined by

$$F(y_1, y_2) \Leftrightarrow \bigwedge_{i \leq n} y_1 \in S_i \leftrightarrow y_2 \in S_i.$$

Let $D = \{d_1, \ldots, d_n\}$ be a set of representatives for all the F-classes contained in S, and observe that for every $f \in \mathrm{Aut}(\mathfrak{C}/D)$, $f(d_i/F) = d_i/F$ and therefore $f(S) = S$. Since S is relatively definable and D-invariant, it is relatively definable with parameters from P.

Now let $(S_\alpha : \alpha \in Ord)$ be an enumeration of the orbit of S under $\mathrm{Aut}(\mathfrak{C})$ and fix an enumeration of P. We will construct an automorphism τ of P which cannot be extended to an automorphism of $\mathfrak{C}$, namely such that $\tau(S) \neq S_\alpha$ for all $\alpha < \kappa$. We do it inductively on α.

For the successor case, assume $\tau_\alpha : P_\alpha \rightarrow P'_\alpha$ is an elementary bijection between subsets $P_\alpha, P'_\alpha \subseteq P$ such that $\tau_\alpha(S \cap P_\alpha) \nsubseteq S_\beta$ for all $\beta < \alpha$.

We extend τ_α to $\tau_{\alpha+1}$ in two steps. In the first step we add, if necessary, a tuple (a, a') to the graph of of τ_α to guarantee that $\tau_{\alpha+1}(S \cap (P_\alpha \cup \{a\})) \nsubseteq S_\alpha$, i.e., to ensure that in the end $\tau(S) \neq S_\alpha$. In the second step we simply add a tuple (b, b') to the graph of τ_α to guarantee that in the end $\mathrm{dom}(\tau) = P$. We describe the steps:

Step 1. If there is no $g \in \mathrm{Aut}(\mathfrak{C})$ extending τ_α such that $g(S) = S_\alpha$, we don't need this step, so let $a = a' = \emptyset$ and go to step 2. So assume there is such g. By hypothesis there is $f \in \mathrm{Aut}(\mathfrak{C}/P_\alpha)$ such that $f(S) \nsubseteq S$, so there is $a \in S$ such that the partial type

$$\mathrm{tp}(a/P_\alpha) \cup \{x \notin S\}$$

is consistent. Since g extends τ_α and $g(S) = S_\alpha$, we know also that the partial type

$$\mathrm{tp}^{\tau_\alpha}(a/P_\alpha) \cup \{x \notin S_\alpha\}$$

is consistent, so let $a' \in P$ be any realization of it.

Step 2. Let b the first element in the enumeration of P which is not in $\mathrm{dom}(\tau_\alpha)$ and

let $b' \in P$ such that $(a', b') \models \mathrm{tp}^{\tau_\alpha}(ab/P_\alpha)$.
Now let $P_{\alpha+1} = P_\alpha \cup \{a, b\}$, $P'_{\alpha+1} = P'_\alpha \cup \{a', b'\}$ and

$$\tau_{\alpha+1} = \tau_\alpha \cup \{(a, a'), (b, b')\},$$

and observe that $\tau_{\alpha+1}(S \cap P_{\alpha+1}) \nsubseteq S_\alpha$.
For the limit case it is easy to see that the union works.
(5) $\to$ (1). We will first see that $\mathrm{tp}(a/P)$ is definable over $\mathrm{dcl}^{\mathrm{eq}}(a) \cap \mathrm{dcl}^{\mathrm{eq}}(P)$. Let $\varphi(x, y)$ be a formula. By stable embeddability,

$$\{b \in P^n :\models \varphi(a, b)\} = \{b \in P^n :\models \psi(y, c)\}$$

for some formula $\psi(y, z) \in L$ and some $c \in P^l$. $\psi(y, c)$ is clearly a definition for $\mathrm{tp}(a/P) \restriction \varphi$. By compactness, there is a 0-definable set $D \supseteq P^n$ such that

$$\{b \in D :\models \varphi(a, b)\} = \{b \in D :\models \psi(y, c)\}.$$

Consider the equivalence relation given by

$$E(z_1, z_2) \Leftrightarrow \forall y \in D(\psi(y, z_1) \leftrightarrow \psi(y, z_2)),$$

and observe that for any $f \in \mathrm{Aut}(\mathfrak{C}/a)$,

$$D \cap \psi(\mathfrak{C}, c) = D \cap \varphi(a, \mathfrak{C}) = D \cap \varphi^f(a, \mathfrak{C}) = D \cap \psi^f(\mathfrak{C}, c) = D \cap \psi(\mathfrak{C}, f(c)),$$

i.e., $f(c/E) = c/E$. This shows that $c/E \in \mathrm{dcl}^{\mathrm{eq}}(a)$, and since $c \in P^l$, we also know that $c/E \in \mathrm{dcl}^{\mathrm{eq}}(P)$. Thus the formula

$$\forall y \in D(\varphi(x, y) \leftrightarrow \exists z(\psi(y, z) \wedge E(z, c)))$$

belongs to $\mathrm{tp}(a/\mathrm{dcl}^{\mathrm{eq}}(a) \cap \mathrm{dcl}^{\mathrm{eq}}(P))$. Since $P^n \subseteq D$, we have that

$$\mathrm{tp}(a/\mathrm{dcl}^{\mathrm{eq}}(a) \cap \mathrm{dcl}^{\mathrm{eq}}(P)) \vdash \mathrm{tp}(a/P).$$

$\square$

2

Resplendent Models and the Lascar group

2.1 Introduction

The presentation of the Lascar group in [13] is done in the framework of an uncountable saturated model N of an arbitrary countable complete first-order theory and a finite subset $A \subseteq N$. It is straightforward to generalize it to any complete first-order theory T and any saturated model N of T with $|N| > |T|$. Thus it always can be constructed working in the monster model $\mathfrak{C}$ of T. Although the details have not been written, it is generally acknowledged that instead of saturated models one can use special models of the right cardinality. For instance, Ziegler observes in [25] that a special model N such that $\mathrm{cf}(|N|) > 2^{|T|}$ is sufficient. The inconvenience of working with saturated models is that for some theories its existence can not be proven without extra set theoretical hypotheses. On the other hand special models do always exist.

We have noticed that there is a more general class of models where the Lascar group naturally arises: the class of $|T|^+$*-resplendent models*. Moreover, the properties of the group of strong automorphisms can be understood more easily working with these models.

The notion of resplendency has been introduced by Barwise and Schlipf in [2]. Poizat in [18] defined and studied the more general notion of κ-resplendency. In Section 2.2 we summarize the main facts. $|T|^+$-resplendent models generalize (in the right cardinality) saturated and special models, and in the case of stable theories they coincide with saturated models. In unstable theories there are many $|T|^+$-resplendent models which are not saturated nor special.

We focuss on the pure abstract group since there is nothing new concerning the topology. The topology of the Lascar group can be explained as in [25] using only the $|T|^+$-saturation of the model and the presence of the pure group. In Section 2.3 we state and prove the main results. Theorem 2.3.14 shows that any $|T|^+$-resplendent model N gives rise to the Lascar group and Theorem 2.3.11 indicates that the group of strong automorphisms over A can be characterized (similarly to what Lascar originally did working with saturated models) as the least *very normal* subgroup of $\mathrm{Aut}(N/A)$, that is, its least normal subgroup closed under a more general conjugation that we call *weak conjugation*. It should be noticed that the methods used in the proofs are quite different. In particular we never use ultraproducts. Finally in section 2.4 we show that these results also hold in the wider class of all $|T|^+$-saturated and strongly $|T|^+$-homogeneous models.

2.2 $|T|^+$-resplendency

Definition 2.2.1. *Let κ be an infinite cardinal number and let M be an $\mathcal{L}$-structure. We say that M is* κ-expandable *if for every language $\mathcal{L}' \supseteq \mathcal{L}$ with $|\mathcal{L}' \smallsetminus \mathcal{L}| < \kappa$, if Σ is a set of sentences of language $\mathcal{L}'$ consistent with* $\mathrm{Th}(M)$*, then there is an $\mathcal{L}'$-expansion of M satisfying Σ. We say that M is* κ-resplendent *if and only if for every $A \subseteq M$ with $|A| < \kappa$, M_A is κ-expandable.*

Here our interest in κ-resplendency resides in the case $\kappa = |T|^+$. Hence we will restrict our attention to this particular case in the next results, some of which can be easily generalized to other cardinal numbers κ.

Proposition 2.2.2.

1. *For every M there is some $|T|^+$-resplendent model $N \succeq M$ such that $|N| \leq |M| + 2^{|T|}$.*
2. *Every saturated model M such that $|M| > |T|$ and every special model M such that* $\mathrm{cf}(|M|) > |T|$ *is $|T|^+$-resplendent.*
3. *$|T|^+$-resplendent models are strongly $|T|^+$-homogeneous and $|T|^+$-saturated.*
4. *If T is stable, every $|T|^+$-resplendent model is saturated.*
5. *If T is unstable, then for every M there is some $|T|^+$-resplendent model $N \succeq M$ of cardinality $|N| \leq |M| + 2^{|T|}$ which is not $|T|^{++}$-saturated.*

Proof. 1 See the proof of Théorème 9.15 in [18]. *2.* Shelah proves in Conclusion I.1.13 of [19] that every saturated model of cardinality $> |T|$ is $|T|^+$-expandable. From this it follows immediately that it is also $|T|^+$-resplendent. The same fact is also proven by Poizat in Théorème 9.17 of [18]. The case of a special model is considered by the first author in [3], where in Proposition 1.2 it is established that every special model of cardinality $> |T|$ is $|T|^+$-expandable. Now, if M is special and $A \subseteq M$ is of cardinality $< \mathrm{cf}(|M|)$, then M_A is still special and hence $|T_A|^+$-expandable. Therefore M is $|T|^+$-expandable if $\mathrm{cf}(|M|) > |T|$. *3.* It is obvious that a $|T|^+$-resplendent model M is $|T|^+$-saturated. To check that it is strongly $|T|^+$-homogeneous, we consider a partial elementary mapping f, $|f| \leq |T|$, with domain and range contained in M, and we show that f can be extended to an automorphism of M. Let $A = \mathrm{dom}(f)$, add a new unary functional symbol F, put $\mathcal{L}' = \mathcal{L} \cup \{F\}$ and note that the set Σ of sentences of $\mathcal{L}'(A \cup f(A))$

1. F is an automorphism.

2. $F(a) = f(a)$ for all $a \in A$.

is consistent with $T_A = \mathrm{Th}(M_A)$. By $|T|^+$-resplendency there is an expansion (M, F^M) of M such that $(M_{A\cup f(A)}, F^M) \models \Sigma$. Clearly, F^M is an automorphism of M extending f. Finally, *4* and *5* are due to Poizat. They follow from Théorème 16.11 and Théorème 14.10 of [18] respectively. The bound on the cardinality of N in *5* can be obtained from the proof given there. $\square$

Corollary 2.2.3. *If T is unstable, for every M, for every cardinal $\kappa \geq |M| + 2^{|T|}$ such that* $\mathrm{cf}(\kappa) > |T|^+$*, there is some nonspecial model $N \succeq M$ of cardinality κ which is $|T|^+$-resplendent.*

Proof. By point *5* of Proposition 2.2.2, there is some $|T|^+$-resplendent $N \succeq M$ of cardinality κ which is not $|T|^{++}$-saturated. Since $\mathrm{cf}(\kappa) > |T|^+$, every special model of cardinality κ is $|T|^{++}$-saturated. Hence N is not special. $\square$

2.3 The Lascar group

Definition 2.3.1. *Let $A \subseteq M$ and $f, g \in \mathrm{Aut}(M/A)$. We say that f, g are* A-conjugate *and we write $f \sim_A g$ if they are conjugate elements of the group* $\mathrm{Aut}(M/A)$*, that is, if $g = \varepsilon \circ f \circ \varepsilon^{-1}$ for some $\varepsilon \in \mathrm{Aut}(M/A)$. We say that they are* weakly A-conjugate *and we write $f \approx_A g$ if for some $N \succeq M$ there are extensions $f \subseteq f' \in$* $\mathrm{Aut}(N/A)$ *and $g \subseteq g' \in$* $\mathrm{Aut}(N/A)$ *such that $f' \sim_A g'$. We say that a subgroup G of* $\mathrm{Aut}(M/A)$ *is* very normal *if it is closed under weakly conjugation, that is if for any $f \in G$ and any $g \in$* $\mathrm{Aut}(M/A)$ *such that $f \approx_A g$ we have $g \in G$. We use $[f]_{\approx_A}$ to denote the $\approx_A$-class of $f \in$* $\mathrm{Aut}(M/A)$*, i.e., $[f]_{\approx_A} = \{g \in \mathrm{Aut}(M/A) : f \approx_A g\}$. This notation should not suggest that weak conjugation is an equivalence relation.*

Remark 2.3.2. *Let $f, g \in Aut(M/A)$. Then $f \approx_A g$ if and only if there are extensions $f \subseteq \bar{f} \in$* $\mathrm{Aut}(\mathfrak{C}/A)$ *and $g \subseteq \bar{g} \in$* $\mathrm{Aut}(\mathfrak{C}/A)$ *such that $\bar{f} \sim_A \bar{g}$.*

Remark 2.3.3.

1. *Very normal subgroups are normal.*

2. *In general $\{\mathrm{id}_M\}$ is not a very normal subgroup of* $\mathrm{Aut}(M/A)$*, as shown in Proposition 2.3.6.*

3. *The intersection of any family of very normal subgroups of* $\mathrm{Aut}(M/A)$ *is again a very normal subgroup.*

Definition 2.3.4. *We denote by* $\Gamma(M/A)$ *the intersection of all very normal subgroups of* $\mathrm{Aut}(M/A)$*, which is again very normal. Note that* $\Gamma(M/A)$ *is a union of* $\approx_A$*-classes and contains* $[\mathrm{id}_M]_{\approx_A}$.

Proposition 2.3.5. *For* $A \subseteq M' \preceq M$, $\mathrm{Aut}(M/M') \subseteq [\mathrm{id}_M]_{\approx_A}$

Proof. Let $f \in \mathrm{Aut}(M/M')$ and expand the language $\mathcal{L}(M)$ by adding three new unary function symbols E, F, G. Let Σ be the set of sentences in the expanded language expressing that

1. F, G, E are A-automorphisms.
2. $G = E^{-1} \circ F \circ E$
3. $G(m) = m$ for all $m \in M$.
4. $f(m) = F(m)$ for all $m \in M$.

If Σ is consistent, we have finished since there is an expansion $\mathfrak{C}'$ of $\mathfrak{C}_M$ which satisfies Σ and then $f \subseteq F^{\mathfrak{C}'} \sim_A G^{\mathfrak{C}'} \supseteq \mathrm{id}_M$, which shows that $f \approx_A \mathrm{id}_M$. To show the consistency of Σ (with T_M) let $a \in M$ be a finite tuple and let us prove that there are $\bar{f}, \bar{g}, \varepsilon \in \mathrm{Aut}(\mathfrak{C}/A)$ such that $\bar{g} = \varepsilon^{-1} \circ \bar{f} \circ \varepsilon$, $\bar{g}(a) = a$ and $\bar{f}(a) = f(a)$. To do this we first check that if $p(x) = \mathrm{tp}(a/A)$, then

$$(*) \quad p(x) \cup \text{“}xa \equiv_A xf(a)\text{” is consistent.}$$

Let $\varphi(x) \in p(x)$. Since $A \subseteq M'$, there is $b \in M'$ such that $\models \varphi(b)$. Since f is the identity in M', $f(b) = b$ and hence $ba \equiv_A bf(a)$. This ensures the consistency. By $(*)$, there is a tuple a' such that $a' \equiv_A a$ and $a'a \equiv_A a'f(a)$. Now choose automorphisms $\bar{f}, \varepsilon \in \mathrm{Aut}(\mathfrak{C}/A)$ with $\varepsilon(a) = a'$ and $\bar{f}(a'a) = a'f(a)$. If $\bar{g} = \varepsilon^{-1} \circ \bar{f} \circ \varepsilon$, it follows that $\bar{g}(a) = a$. □

Proposition 2.3.6. *If* N *is* $|T|^+$*-resplendent and* $A \subseteq N$ *is of cardinality at most* $|T|$*, then* $|[\mathrm{id}_N]_{\approx_A}| \geq |N|$*. In particular* $\{\mathrm{id}_N\}$ *is not a very normal subgroup of* $\mathrm{Aut}(N/A)$.

Proof. Choose an elementary submodel $M \preceq N$ containing A of cardinality $\leq |T|$ and choose a nonalgebraic type $p(x) \in S(M)$. In the monster model we may find a proper class P of realizations of $p(x)$. Let F be a new binary functional symbol, let $\mathcal{L}' = \mathcal{L} \cup \{F\}$, and look at the following set Σ of sentences of $\mathcal{L}'(M)$:

1. For all x, the mapping $y \mapsto F(x, y)$ is an automorphism.

2. $\forall x F(x, a) = a$ for all $a \in M$.

3. $\forall xy(x \neq y \to \exists z\, F(x, z) \neq F(y, z))$.

The consistency of this set of sentences with $\mathrm{Th}(N_M)$ follows from the fact that in the monster model we may find for each two different $a, b \in P$ an automorphism $f \in \mathrm{Aut}(\mathfrak{C}/M)$ with $f(a) = b$. Now, by $|T|^+$-resplendency of N there is an expansion (N, F^N) of N such that $(N_M, F^N) \models \Sigma$. For each $a \in N$ we get a different automorphism $f_a \in \mathrm{Aut}(N/M)$ defined by $f_a(b) = F^N(a, b)$. By Proposition 2.3.5 $f_a \approx_A \mathrm{id}_N$ for all $a \in N$. $\square$

Definition 2.3.7. *Let $A \subseteq M$. The group of all* strong automorphisms over A *is the subgroup* $\mathrm{Autf}(M/A)$ *of* $\mathrm{Aut}(M/A)$ *generated by the union of all subgroups* $\mathrm{Aut}(M/M')$ *for all possible M' such that $A \subseteq M' \preceq M$.*

Remark 2.3.8. *It is easy to check that* $\mathrm{Autf}(M/A)$ *is a normal subgroup of* $\mathrm{Aut}(M/A)$*. From Proposition 2.3.5 it follows that it is also a subgroup of* $\Gamma(M/A)$*.*

Proposition 2.3.9. *Assume N is $|T|^+$-resplendent, $A \subseteq N$, $|A| \leq |T|$, $f \in \mathrm{Aut}(N/A)$, and $f \subseteq \bar{f} \in \mathrm{Aut}(\mathfrak{C}/A)$. Then $f \in \mathrm{Autf}(N/A)$ if and only if $\bar{f} \in \mathrm{Autf}(\mathfrak{C}/A)$.*

Proof. Let $f = f_1 \circ \ldots \circ f_n$ where for each i, $f_i \in \mathrm{Aut}(N/A)$ is the identity in a submodel $M_i \preceq N$ containing A. If we take extensions $f_i \subseteq \bar{f}_i \in \mathrm{Aut}(\mathfrak{C}/A)$ and consider $g = \bar{f}_1 \circ \ldots \circ \bar{f}_n$, we see that $\bar{f} \circ g^{-1} \in \mathrm{Aut}(\mathfrak{C}/N)$, $\bar{f}_i \in \mathrm{Aut}(\mathfrak{C}/M_i)$, and $\bar{f} = (\bar{f} \circ g^{-1}) \circ \bar{f}_1 \circ \ldots \circ \bar{f}_n$, which shows that $\bar{f} \in \mathrm{Autf}(\mathfrak{C}/A)$.

Now assume $\bar{f} \in \mathrm{Autf}(\mathfrak{C}/A)$ and choose $\bar{f}_1, \ldots, \bar{f}_n \in \mathrm{Aut}(\mathfrak{C}/A)$ such that $\bar{f} = \bar{f}_1 \circ \ldots \circ \bar{f}_n$ and $\bar{f}_i \in \mathrm{Aut}(\mathfrak{C}/M_i)$ for some $M_i \supseteq A$. Choose a model $M \preceq N$ with $A \subseteq M$ and $|M| \leq |T|$ closed under f and f^{-1}. Enlarge the language $\mathcal{L}(M)$ by adding unary function symbols $F_1, \ldots, F_n, G$ and unary predicates $U_1, \ldots, U_n$ and let Σ be the set of sentences expressing

1. $F_1, \ldots, F_n, G$ are A-automorphisms.

2. $U_1, \ldots, U_n$ are elementary submodels containing A.

3. $F_i \restriction U_i = \mathrm{id}_{U_i}$ for all $i = 1, \ldots, n$.

4. $G = F_1 \circ \ldots \circ F_n$

5. $G(m) = f(m)$ for all $m \in M$.

Σ is consistent. By $|T|^+$-resplendency there is an expansion of N_M which satisfies Σ. This gives us some $g \in \mathrm{Autf}(N/A)$ such that $f \restriction M = g \restriction M$. Hence $f \circ g^{-1} \in \mathrm{Aut}(N/M) \subseteq \mathrm{Autf}(N/A)$ and therefore $f \in \mathrm{Autf}(N/A)$. $\square$

Remark 2.3.10. *Note that the above proposition also holds if instead of the monster model $\mathfrak{C}$ we choose an elementary extension $N' \succeq N$ which is $|N|^+$-resplendent.*

Theorem 2.3.11. *For any $|T|^+$-resplendent model N and any $A \subseteq N$ such that $|A| \leq |T|$, $\mathrm{Autf}(N/A) = \Gamma(N/A)$.*

Proof. By Proposition 2.3.5 we know that $\mathrm{Autf}(N/A) \subseteq \langle [\mathrm{id}_N]_{\approx_A} \rangle \subseteq \Gamma(N/A)$. To show that $\Gamma(N/A) \subseteq \mathrm{Autf}(N/A)$ it is enough to check that $\mathrm{Autf}(N/A)$ is a very normal subset of $\mathrm{Aut}(N/A)$. Let $f \in \mathrm{Autf}(N/A)$ and let $g \in \mathrm{Aut}(N/A)$ be such that $f \approx_A g$. For some $\bar{f}, \bar{g} \in \mathrm{Aut}(\mathfrak{C}/A)$ extending f, g respectively we have that $\bar{f} \sim_A \bar{g}$. By Proposition 2.3.9, $\bar{f} \in \mathrm{Autf}(\mathfrak{C}/A)$. Since $\mathrm{Autf}(\mathfrak{C}/A)$ is a normal subgroup of $\mathrm{Aut}(\mathfrak{C}/A)$, $\bar{g} \in \mathrm{Autf}(\mathfrak{C}/A)$. Again by Proposition 2.3.9, $g \in \mathrm{Autf}(N/A)$. $\square$

Corollary 2.3.12. *For any $|T|^+$-resplendent model N and any $A \subseteq N$ such that $|A| \leq |T|$, $\langle [\mathrm{id}_N]_{\approx_A} \rangle = \Gamma(N/A)$.*

Proof. By Proposition 2.3.5 and Theorem 2.3.11. $\square$

Remark 2.3.13.

1. *It is easy to check (see Proposition 34 in [13]) that an automorphism $f \in \mathrm{Aut}(N/A)$ is in $[\mathrm{id}_N]_{\approx_A}$ if and only if $N \equiv_M f(N)$ for some model $M \supseteq A$. In other words, for some model $M \supseteq A$, f has an extension in $\mathrm{Aut}(\mathfrak{C}/M)$.*

2. *Lascar in [13] pointed out that for stable T, $[\mathrm{id}_N]_{\approx_A}$ itself is a group. In a stable theory, $\mathrm{Autf}(N/A) = \mathrm{Aut}(N/\mathrm{acl}^{\mathrm{eq}}(A))$ and it suffices to take for the model M in point 1 a realization of the nonforking extension of $\mathrm{tp}(N/\mathrm{acl}^{\mathrm{eq}}(A))$ over N. Hence $\mathrm{Autf}(N/A) = [\mathrm{id}_N]_{\approx_A}$ for any $|T|^+$-resplendent model N of a stable theory T.*

3. *If T is o-minimal, then for every model $N \supseteq A$, $\mathrm{Aut}(N/A) = [\mathrm{id}_N]_{\approx_A}$. This follows from the proof of Lemma 24 in [25] where it is shown that for every $f \in \mathrm{Aut}(N/A)$ there is some extension $\bar{f} \in \mathrm{Aut}(\mathfrak{C}/A)$ of f which is the identity on some model $M \supseteq A$.*

Theorem 2.3.14. *For any $|T|^+$-resplendent model N and any $A \subseteq N$ such that $|A| \leq |T|$, $\mathrm{Aut}(N/A)/\mathrm{Autf}(N/A)$ is independent of the choice of N.*

Proof. We prove that $\mathrm{Aut}(N/A)/\mathrm{Autf}(N/A) \cong \mathrm{Aut}(\mathfrak{C}/A)/\mathrm{Autf}(\mathfrak{C}/A)$ For this, we define a mapping $\delta : \mathrm{Aut}(N/A) \to \mathrm{Aut}(\mathfrak{C}/A)/\mathrm{Autf}(\mathfrak{C}/A)$ choosing for any $f \in \mathrm{Aut}(N/A)$ an arbitrary extension $f \subseteq \bar{f} \in \mathrm{Aut}(\mathfrak{C}/A)$ and putting $\delta(f) = \bar{f}\mathrm{Autf}(\mathfrak{C}/A)$. It is clearly well defined an it is a group homomorphism. From Proposition 2.3.9 it follows that its kernel is $\mathrm{Autf}(N/A)$. We finish the proof by showing that δ is onto. Let $g \in \mathrm{Aut}(\mathfrak{C}/A)$. We seek some $f \in \mathrm{Aut}(N/A)$ such that $\delta(f) = g\mathrm{Autf}(\mathfrak{C}/A)$.

Choose a submodel $M \preceq N$ such that $A \subseteq M$ and $|M| \leq |T|$. By $|T|^+$-saturation of N we may find in N a realization M' of $\mathrm{tp}(g(M)/M)$. Then $g(M) \equiv_M M'$ and there is some $h \in \mathrm{Aut}(\mathfrak{C}/M)$ such that $h(g(M)) = M'$. M' is an elementary submodel of N containing A and $(h \circ g) \upharpoonright M$ is an A-isomorphism between M and M'. We enlarge the language $\mathcal{L}(M \cup M')$ by adding an unary function symbol F. Let Σ be the set of sentences expressing

1. F is an A-automorphism.
2. $F(a) = h(g(a))$ for all $a \in M$.

It is consistent and by $|T|^+$-resplendency there is some $f \in \mathrm{Aut}(N/A)$ such that $f \upharpoonright M = h \circ g \upharpoonright M$. Let $\bar{f} \in \mathrm{Aut}(\mathfrak{C}/A)$ be an arbitrary extension of f. Then $h \circ g \circ \bar{f}^{-1} \in \mathrm{Aut}(\mathfrak{C}/M') \subseteq \mathrm{Autf}(\mathfrak{C}/A)$ and $h \in \mathrm{Autf}(\mathfrak{C}/A)$. Therefore $g \circ \bar{f}^{-1} \in \mathrm{Autf}(\mathfrak{C}/A)$, that is, $\delta(f) = \bar{f}\mathrm{Autf}(\mathfrak{C}/A) = g\mathrm{Autf}(\mathfrak{C}/A)$. □

2.4 $|T|^+$-saturated strongly $|T|^+$-homogeneous models

Theorems 2.3.11 and 2.3.14 hold for a class of models strictly wider than the class of all $|T|^+$-resplendent models. The arguments given so far can be refined to show that they are also true for $|T|^+$-saturated strongly $|T|^+$-homogeneous models. As pointed out in point *3* of Proposition 2.2.2, all $|T|^+$-resplendents models are $|T|^+$-saturated and strongly $|T|^+$-homogeneous. However it is easy to find nonsaturated models of stable theories (even of Morley rank 2) which are $|T|^+$-saturated and strongly $|T|^+$-homogeneous. After point *4* of Proposition 2.2.2 it is clear that these models are not $|T|^+$-resplendent.

We will now indicate shortly how the proofs given in the previous sections can be modified to obtain this more general result. One key point is that theorems 2.3.11 and 2.3.14 depend basically on what we call the *extension property.*

Definition 2.4.1. *Let $A \subseteq M$. We say that M has the* extension property over A *if every $f \in \mathrm{Aut}(M/A)$ which has an extension $\bar{f} \supseteq f$ in $\mathrm{Autf}(\mathfrak{C}/A)$ is already in* $\mathrm{Autf}(M/A)$.

Remark 2.4.2. *It is always true that every $\bar{f} \in \mathrm{Aut}(\mathfrak{C}/A)$ extending some $f \in \mathrm{Autf}(M/A)$ is strong.*

Proposition 2.3.9 shows that all $|T|^+$-resplendent models have the extension property over small subsets. We show that this is also the case for $|T|^+$-saturated strongly $|T|^+$-homogeneous models. Our proof uses the same idea as the one presented by Ziegler in the proof of Corollary 3 in [25].

Proposition 2.4.3. *Assume N is $|T|^+$-saturated and strongly $|T|^+$-homogeneous. Then for each $A \subseteq N$ such that $|A| \leq |T|$, N has the extension property over A.*

Proof. Let $f \subseteq \bar{f} \in \mathrm{Autf}(\mathfrak{C}/A)$ and choose $\bar{f}_1, \ldots, \bar{f}_n$ such that $\bar{f} = \bar{f}_1 \circ \ldots \circ \bar{f}_n$ and $\bar{f}_i \in \mathrm{Aut}(\mathfrak{C}/N_i)$ for some $N_i \supseteq A$. We may assume that $|N_i| \leq |T|$ for each i. Choose a model $M \preceq N$ with $A \subseteq M$ and $|M| \leq |T|$. Let $M_0 = M$ and $M_{i+1} = \bar{f}_{i+1}(M_i)$ for $i = 0, \ldots, n-1$. Observe that

$$M = M_0 \equiv_{N_1} M_1 \equiv_{N_2} M_2 \ldots \equiv_{N_{n-1}} M_{n-1} \equiv_{N_n} M_n = \bar{f}(M) = f(M)$$

By using $|T|^+$-saturation of N, choose now models $N'_i \preceq N$ and $M'_i \preceq N$ such that

$$M_0, \ldots, M_n, N_1 \ldots, N_n \equiv_{Mf(M)} M'_0, \ldots, M'_n, N'_1 \ldots, N'_n$$

Then $A \subseteq N'_i$ and

$$M = M'_0 \equiv_{N'_1} M'_1 \equiv_{N'_2} M'_2 \ldots \equiv_{N'_{n-1}} M'_{n-1} \equiv_{N'_n} M'_n = f(M).$$

By $|T|^+$-strong homogeneity of N we may now find $g_{i+1} \in \mathrm{Aut}(N/N'_i)$ such that $g_{i+1}(M'_i) = M'_{i+1}$ for $i = 0, \ldots, n-1$. Let $g = g_1 \circ \ldots \circ g_n$. Then $g \in \mathrm{Autf}(N/A)$ and $g \restriction M = f \restriction M$. Hence $g \circ f^{-1} \in \mathrm{Aut}(N/M) \subseteq \mathrm{Autf}(N/A)$ and hence $f \in \mathrm{Autf}(N/A)$. $\square$

Theorem 2.4.4. *For any $|T|^+$-saturated and strongly $|T|^+$-homogeneous model N and any $A \subseteq N$ such that $|A| \leq |T|$, $\mathrm{Autf}(N/A) = \Gamma(N/A) = \langle [\mathrm{id}_N]_{\approx_A} \rangle$ and $\mathrm{Aut}(N/A)/\mathrm{Autf}(N/A)$ is independent of the choice of N.*

Proof. For the first assertion, observe that the proof given in Theorem 2.3.9 shows in fact that the extension property over $A \subseteq N$ is enough to get $\mathrm{Autf}(N/A) = \Gamma(N/A) = \langle [\mathrm{id}_N]_{\approx_A} \rangle$, and then use Proposition 2.4.3. The proof of the existence of an isomorphism between $\mathrm{Aut}(N/A)/\mathrm{Autf}(N/A)$ and $\mathrm{Aut}(\mathfrak{C}/A)/\mathrm{Autf}(\mathfrak{C}/A)$ is a slight modification of the proof given for Theorem 2.3.14. To check that δ is onto, instead of enlarging the language by adding F, use $|T|^+$-strong homogeneity of N to obtain some $f \in \mathrm{Aut}(N/A)$ such that $f \restriction M = h \circ g \restriction M$. Let $\bar{f} \in \mathrm{Aut}(\mathfrak{C}/A)$ be an arbitrary extension of f. Then $h \circ g \circ \bar{f}^{-1} \in \mathrm{Aut}(\mathfrak{C}/M') \subseteq \mathrm{Autf}(\mathfrak{C}/A)$ and $h \in \mathrm{Autf}(\mathfrak{C}/A)$. Therefore $g \circ \bar{f}^{-1} \in \mathrm{Autf}(\mathfrak{C}/A)$, that is, $\delta(f) = \bar{f}\mathrm{Autf}(\mathfrak{C}/A) = g\mathrm{Autf}(\mathfrak{C}/A)$. $\square$

3

G-compactness

As mentioned in section 1.1, the notion of G-compactness was introduced by Lascar in [13], and different characterizations where given later in [11], [14], [25] and [16].

In this chapter we are interested in two main problems around this notion. First we tackle the question of wether G-compactness is preserved after adding parameters to a G-compact theory. We answer this question negatively in section 3.2 , providing several examples of a G-compact (over $\emptyset$) theory T and a set of parameters A such that T is not G-compact over A. In section 3.3 we present a new proof of Newelski's Corollary 1.8 in [16], which states that a type-definable Lascar strong type has finite diameter. For this purpose we use some techniques introduced also by Newelski in [17]. In the next section we present the first examples of non G-compact theories that appeared in [4]. Throughout this chapter, d and d_A will denote the distance introduced in section 1.1.

3.1 Non-G-compact theories

The following examples were first exhibited in [4], but we work with a relational language. Let C be a circle with perimeter 1. Fix a natural number $n > 1$. Let C_n be the $\mathcal{L}_n$-structure with universe C in the language $\mathcal{L}_n = \{B_n, L_n, R_n^m, S_n^m : 1 \leq m \leq n\}$ where:

(a) $B_n(a, b, c)$ holds if a, b, c are different points of C_n and if, starting from a and going around the circle clockwise, b appears sooner than c.

(b) $L_n(a, b)$ holds if the distance between a and b is shorter going clockwise from a to b than the other way around.

(c) For each $1 \leq m \leq n$, $R_n^m(a, b)$ holds if the length of the shortest arch joining a and b is less or equal than $m/2n$.

(d) For each $1 \leq m \leq n$, $S_n^m(a, b)$ holds if the length of the shortest arch joining a and b is strictly less than $m/2n$.

We fix some $n > 1$ and omit the subindexes for convenience in notation. It is easy to see that for each $m \in \{1, \ldots, n\}$, R^m is definable in terms of R^1 (from now on just R) and the same for S^m in terms of S. We can also see that $R(x, y)$ can be defined in terms of B and S by the formula

$$\forall z((B(x, z, y)) \to S(x, z)) \vee \forall z((B(y, z, x)) \to S(y, z)),$$

and also that $L(x, y)$ can be defined by the formula

$$S^n(x, y) \wedge \forall z(B(x, z, y) \to S^n(x, z)).$$

Now, the clockwise rotation by π/n radians inside C_n, $g_n : C_n \to C_n$ (or just g), is a bijection definable in the language $\mathcal{L}_n$ by the formula

$$g(x) = y \leftrightarrow L(x, y) \wedge R(x, y) \wedge \neg S(x, y).$$

Observe also that S (and therefore L, S^m and R^m for $1 \leq m \leq n$) is definable with the symbols B and g by the formula

$$B(x, y, g(x)) \vee B(y, x, g(y)).$$

From this we know that the following remark is true.

Remark 3.1.1. *Fix $n > 1$. Let $\mathcal{L}_n^g = \{B_n, g_n\}$ and consider C_n as an $\mathcal{L}_n^g$-structure where B_n is interpreted as before and the function symbol g_n is interpreted as the clockwise rotation by π/n radians. Then for any two finite tuples $\bar{a}, \bar{b} \in C_n$,*

$$\mathrm{tp}_{\mathcal{L}_n}(\bar{a}) = \mathrm{tp}_{\mathcal{L}_n}(\bar{b}) \Leftrightarrow \mathrm{tp}_{\mathcal{L}_n^g}(\bar{a}) = \mathrm{tp}_{\mathcal{L}_n^g}(\bar{b})$$

We give axioms for $\mathrm{Th}(C_n)$ in the language $\mathcal{L}_n^g$ for $n > 1$:

A1. For all x, $\{(y, z) : B(x, y, z)\}$ is a dense strict linear order without endpoints.

A2. $\forall xyz(B(x, y, z) \leftrightarrow B(y, z, x))$.

A3. $\forall x(g^{2n}(x) = x)$.

A4. $\forall x B(x, g^i(x), g^j(x))$ for any $0 < i < j < 2n$.

A5. $\forall xyz(B(x, y, z) \leftrightarrow B(g(x), g(y), g(z)))$.

Proposition 3.1.2. *The theory given by the previous axioms is complete, ω-categorical and has elimination of quantifiers in the language $\mathcal{L}_n^g$.*

Proof. 1. We do back-and-forth with the partial isomorphisms between finitely generated (g-closed) sets. Let f be one of such isomorphisms and let a be such that $a \notin \mathrm{dom}(f)$. If f is the empty function, choose any b and let $f'(a) = b$ and for each $1 \leq i \leq 2n-1$, $f'(g^i(a)) = g^i(b)$. By axiom A4, f' is an isomorphism between finitely generated g-closed sets extending f. Now assume f is not the empty function. Making use of he axioms, let

$$a_1^0, a_2^0, \ldots, a_m^0, a_1^1, a_2^1 \ldots, a_m^1, \ldots, a_1^{2n-1}, a_2^{2n-1}, \ldots, a_m^{2n-1}$$

be an enumeration of $\mathrm{dom}(f)$ such that:

1. It's ordered with respect to the order induced by a.
2. $a_j^i = g(a_j^{i-1})$ for all $i \in \{1, \ldots, 2n-1\}$ and $j \in \{1, \ldots, m\}$.

From these assumptions and axiom A2, we know that $B(a_m^{2n-1}, a, a_1^0)$. And using axiom A5 we also know that $B(a_m^{i-1}, g^i(a), a_1^i)$ for each $i \in \{1, \ldots, 2n-1\}$.
Now, by axiom A1, choose an element b such that $B(b_m^{2n-1}, b, b_1^0)$, where $b_j^i = f(a_j^i)$, and extend f to f' as follows. Let $f'(a) = b$ and for each $1 \leq i \leq 2n-1$, let $f'(g^i(a)) = g_n^i(b)$. It's then clear that f' is an isomorphism with respect to B between

$$\boldsymbol{a}, a_1^0, \ldots, a_m^0, \boldsymbol{g(a)}, a_1^1, \ldots, a_m^1, \ldots, a_m^{2n-2}, \boldsymbol{g^{2n-1}(a)}, a_1^{2n-1}, \ldots, a_m^{2n-1}$$

and

$$\boldsymbol{b}, b_1^0, \ldots, b_m^0, \boldsymbol{g(b)}, b_1^1, \ldots, b_m^1, \ldots, b_m^{2n-2}, \boldsymbol{g^{2n-1}(b)}, b_1^{2n-1}, \ldots, b_m^{2n-1}.$$

To see that f' is also an isomorphism with respect to g, it's enough to observe that $g^i(a) = g^j(a)$ if and only if $g^i(b) = g^j(b)$ for every $i, j \in \{1, \ldots, 2n\}$ and that

$$\mathrm{dom}(f) \cap \left\{g^i(a) : i < 2n\right\} = \mathrm{rng}(f) \cap \left\{g^i(b) : i < 2n\right\} = \emptyset$$

□

Proposition 3.1.3. *$X \subseteq C_n$ is an elementary substructure of C_n (as an $\mathcal{L}_n^g$-structure) if and only if it is closed under g_n and X is dense.*

Proof. From left to right it's clear. For the other direction, observe that the back and forth in the previous proposition can be done between X and C_n. □

Lemma 3.1.4. *For any $n > 1$ and $a, b, c \in C_n$, $\mathrm{qftp}_{\mathcal{L}_n}(a, b, c)$ detrmines $\mathrm{qftp}_{\mathcal{L}_n}(a, b, g(c))$.*

Proof. Let a, b, c be different elements of C_n and assume $B(a, b, c)$. Observe that

1. $g(c) = a \Leftrightarrow L(c, a) \wedge R(c, a) \wedge \neg S(c, a)$.

2. $B(a, b, g(c)) \Leftrightarrow g(c) \neq a \wedge g(c) \neq b \wedge ((S(a, c) \wedge L(c, a)) \rightarrow (L(a, b) \wedge S(c, b)))$.

3. $B(a, g(c), b) \Leftrightarrow g(c) \neq a \wedge g(c) \neq b \wedge \neg B(a, b, g(c))$.

4. $L(a, g(c)) \Leftrightarrow (L(a, c) \wedge S^{n-1}(a, c)) \vee (L(c, a) \wedge S(a, c))$.

5. $L(g(c), a) \Leftrightarrow (L(c, a) \wedge \neg R(c, a)) \vee (\neg L(c, a) \wedge R^{n-1}(c, a))$.

6. $R(a, g(c)) \Leftrightarrow (L(c, a) \wedge R^2(a, c)) \vee a = c$

7. $R^m(a, g(c)) \Leftrightarrow (L(a, c) \wedge R^{m-1}(a, c)) \vee (L(c, a) \wedge R^{m+1}(a, c)) \vee a = c$, for $m \in \{2, \ldots n-1\}$.

And in a similar way for S^m and b instead of a. For the case where two of the three elements are equal, just consider points 1.,4.,5.,6. and 7. in the previous list. □

Lemma 3.1.5. *For any $n > 0$ and two finite tuples $\bar{a}, \bar{b}$ of C_n,*

$$\mathrm{qftp}_{\mathcal{L}_n}(\bar{a}) = \mathrm{qftp}_{\mathcal{L}_n}(\bar{b}) \Leftrightarrow \mathrm{qftp}_{\mathcal{L}_n^g}(\bar{a}) = \mathrm{qftp}_{\mathcal{L}_n^g}(\bar{b})$$

Proof. From right to left it's clear by remark 3.1.1 and the elimination of quantifiers in the language $\mathcal{L}_n^g$ (proposition 3.1.2 (1.)). For the other direction, let $\bar{a} = a_1, \ldots a_m$ and $\bar{b} = b_1, \ldots, b_m$ be two finite tuples of C_n with the same quantifier-free type in the language $\mathcal{L}_n$. It suffices to check the following statements. Let $i, j, k \in \{1, \ldots, 2n\}$ and $r, s, t \in \{1, \ldots m\}$.

1. $g^i(a_r) = g^j(a_s)$ if and only if $g^i(b_r) = g^j(b_s)$.

2. $B(g^i(a_r), g^j(a_s), g^k(a_t))$ if and only if $B(g^i(b_r), g^j(b_s), g^k(b_t))$.

Both of them follow applying lemma 3.1.4 a finite number of times. □

Corollary 3.1.6. *The following are true.*

1. $\mathrm{Th}(C_n)$ *is ω-categorical and has elimination of quantifiers in the language $\mathcal{L}_n$.*

2. *$X \subseteq C_n$ is an elementary substructure of C_n (as an $\mathcal{L}_n$-structure) if and only if it is closed under clockwise rotation by π/n radians and X is dense.*

Proof. By lemma 3.1.5, proposition 3.1.2 and remark 3.1.1, we know that for any two finite tuples $\bar{a}, \bar{b} \in C_n$,

$$\begin{aligned} \mathrm{qftp}_{\mathcal{L}_n}(\bar{a}) = \mathrm{qftp}_{\mathcal{L}_n}(\bar{b}) \quad &\Leftrightarrow \quad \mathrm{qftp}_{\mathcal{L}_n^g}(\bar{a}) = \mathrm{qftp}_{\mathcal{L}_n^g}(\bar{b}) \\ &\Leftrightarrow \quad \mathrm{tp}_{\mathcal{L}_n^g}(\bar{a}) = \mathrm{tp}_{\mathcal{L}_n^g}(\bar{b}) \\ &\Leftrightarrow \quad \mathrm{tp}_{\mathcal{L}_n}(\bar{a}) = \mathrm{tp}_{\mathcal{L}_n}(\bar{b}). \end{aligned}$$

2. follows from proposition 3.1.3 and remark 3.1.1. □

Proposition 3.1.7. *If $a, b \in C_n$, then $S(a, b)$ holds iff a, b have the same type over an elementary substructure of C_n.*

Proof. Suppose $S(a, b)$. Without loss of generality assume $L(a, b)$. Let $I_{a,b} = \{a, b\} \cup \{x : B_n(a, x, b)\}$ and let X be the substructure obtained by removing $I_{a,b}$ and its rotations by $k\pi/n$ radians ($k = 1, \ldots 2n - 1$) from C_n. By proposition 3.1.6 (2.), X is an elementary substructure of C_n and $a \equiv_X b$.
Now suppose that $\neg S_n(a, b)$. Let N be an elementary substructure of C_n and let $c \in N$. Recall that N is closed under g, the clockwise rotation by π/n radians. In case a or b are in the orbit of c under g, it's clear that $a \not\equiv_N b$. Otherwise, there are $m_1, m_2 < 2n$ such that $B(a, g^{m_1}(c), b)$ and $B(b, g^{m_2}(c), a)$. Then $B(g^{m_1}(c), b, g^{m_2}(c))$, but $\neg B(g^{m_1}(c), a, g^{m_2}(c))$, showing that $a \not\equiv_N b$. $\square$

Corollary 3.1.8. *Any two elements of C_n have the same Lascar strong type but $diam(C_n) \geq n$.*

Two structures

Consider the following two structures built from the circles C_n's above (we assume that they're saturated).

1. $\bigsqcup_{n>1} C_n$: The coproduct, or disjoint sum. It's just the family of structures $(C_n : n > 1)$ considered as a many-sorted structure. The n-th sort corresponds to C_n equipped with all its $\mathcal{L}_n$-structure. No additional relation or function symbols.

2. $C = \prod_{n>1} C_n$: The product. An element of its universe is a mapping c defined on $\omega \setminus \{0, 1\}$ and such that $c(n) \in C_n$ for every $n > 1$. Its language is $\mathcal{L} = \bigcup_{n>1} \mathcal{L}'_n$, where $\mathcal{L}'_n = \mathcal{L}_n \cup \{E_n\}$. For any $c, c' \in C$, $n > 1, m \leq n$, we say that $R_n^m(c, c')$ holds if and only if $R_n^m(c(n), c'(n))$ holds in C_n, and so for B_n, L_n and S_n^m. E_n is a binary relation symbol interpreted as follows:

$$E_n(c, c') \Leftrightarrow c(n) = c'(n)$$

   E_n is an equivalence relation and we can recover the circle C_n with all its $\mathcal{L}_n$-structure taking the quotient C/E_n.

   Let $E = \bigcap_{n>1} E_n$ and observe that for every $c \in C$, $C/E = \{c\}$. Adjoining a suitable large number of new elements to each E-class one gets a saturated elementary extension C^* of C. Note that in passing to C^* no new elements were added to any of the C_n's.

Using back-and-forth arguments we can easily see that both $Th\left(\bigsqcup_{n>0} C_n\right)$ and $\mathrm{Th}(C^*)$ have elimination of quantifiers in their respective languages.

Remark 3.1.9. *1.* $\mathrm{Aut}\left(\bigsqcup_{n>1} C_n\right) = \prod_{n>1} \mathrm{Aut}(C_n)$.

2. The canonical map from $\mathrm{Aut}(C^*)$ *to* $\mathrm{Aut}\left(\bigsqcup_{n>1} C_n\right)$ *is surjective.*

Proposition 3.1.10. $\overset{\mathrm{KP}}{\equiv}$ *is trivial for elements in* C^*.

Proof. By definition, $a \overset{\mathrm{KP}}{\equiv} b$ if and only if a and b are equivalent under any type-definable bounded equivalence relation, and we know that any such relation can be defined by a conjunction $\bigwedge_{n<\omega} \theta_n(x,y)$ of thick formulas such that $\theta_{n+1}^2 \subseteq \theta_n$ for every $n < \omega$, i.e., $\theta_{n+1}(x,y) \wedge \theta_{n+1}(y,z) \vdash \theta_n(x,z)$.

Assume, searching for a contradiction, that $\neg\theta_0(a,b)$ for a formula $\theta_0(x,y)$ of some such family and elements $a, b \in C^*$. Since θ_0 is reflexive, $a \neq b$. Let $\mathcal{L}'_2 \cup \dots \mathcal{L}'_N$ be the language of θ_0 and choose n such that $N < 2^n$. Observe that

$$\theta_n^{2^n} \subseteq \theta_{n-1}^{2^{n-1}} \cdots \subseteq \theta_2^4 \subseteq \theta_1^2 \subseteq \theta_0.$$

By quantifier elimination, the formula $\theta_n(x,y)$ is of the form

$$\theta_n(x,y) = \bigvee_{i=1}^{l} \sigma_i$$

for some $l < \omega$, where each σ_i is a conjunction of some of the relations $\{=, L_k, R_k^m, S_k^m, E_k : 1 < k \leq N, 1 \leq m \leq k\}$ and their negations.

Claim 1. At least one of the σ_i's is not negative in any of the relations R_k^m $(1 < k \leq N, 1 \leq m \leq k)$. (This means that at least in one conjunction there is no lower bound for the distance between any two comparable coordinates of x and y).

Proof. Otherwise, in each σ_i there is a term $\neg R_{k_i}^{m_i}$. But the relation $\bigwedge_{i=1}^{l} R_{k_i}^{m_i}$ is thick, since it is impossible to find an infinite antichain for it, i.e. infinitely many elements such that their k_i-th projections are at at distance $> m_i/k_i$. Since

$$\theta_n(x,y) = \bigvee_{i=1}^{l} \sigma_i \vdash \bigvee_{i=1}^{l} \neg R_{k_i}^{m_i},$$

then $\bigwedge_{i=1}^{l} R_{k_i}^{m_i} \vdash \neg\theta_n(x,y)$. This implies that $\neg\theta_n(x,y)$ is thick and $\theta_n(x,y)$ is not, which is a contradiction. $\diamond$

Now let $I = \{i \leq N : E_i(a,b)\}$ and $J = \{i \leq N : \neg E_i(a,b)\}$. We write $\theta_n(x,y) = \rho_n(x,y) \vee \nu_n(x,y)$, where $\rho_n(x,y)$ is the disjunction of the conjunctions which are not negative in any of the relations R_k^m, and $\nu_n(x,y)$ is the disjunction of the remaining ones.

Claim 2. $\rho_n(x,y) \nvdash \bigvee_{i\in I} \neg E_i(x,y) \vee \bigvee_{j\in J} E_j(x,y) \vee x=y$.

Proof. Otherwise, choose in each conjunction of $\nu_n(x,y)$ a relation $R_{k_i}^{m_i}$ appearing negatively (say for $i=1,\dots l'$) and observe that

$$\bigwedge_{i\in I} E_i(x,y) \wedge \bigwedge_{j\in J} \neg E_j(x,y) \wedge x \neq y \wedge \bigwedge_{i=1}^{l'} R_{k_i}^{m_i} \vdash \neg\rho_n(x,y) \wedge \neg\nu_n(x,y) \vdash \neg\theta_n(x,y).$$

This implies that we can find an infinite sequence $(a_i : i < \omega)$ of different elements in C^* such that $\neg\theta_n(a_i,a_j)$, whenever $i<j$. Therefore, $\theta_n(x,y)$ is not thick, and we have again a contradiction. $\diamond$

Since $\rho_n(x,y) \wedge \bigwedge_{i\in I} E_i(x,y) \wedge \bigwedge_{j\in J} \neg E_j(x,y) \wedge x \neq y$ is consistent, we can find elements $a^*, b^* \in C^*$ such that $\rho_n(a^*,b^*)$ and such that $E_i(a^*,b^*)$ if and only if $E_i(a,b)$ for $i=1,\dots,N$. We can also find distinct elements a', b' such that

- $E_i(a'a)$ and $E_i(b',b)$ for $i=2,\dots N$.
- $E_i(a',a^*)$ and $E_i(b`,b^*)$ for $i>N$.

Claim 3. $\theta_n^{2^n}(a',b')$.

If the claim is true, then $\theta_0(a',b')$. By the choice of the E_i-classes and elimination of quantifiers, the pair a',b' satisfies the same formulas of $\mathcal{L}_1 \cup \dots \cup \mathcal{L}_N$. This implies that $\theta_0(a,b)$, which gives us the desired contradiction.

Proof Claim 3. It is enough to find $a_1,\dots,a_{2^n+1}$ such that $a_i=a'$, $a_{2^n+1}=b'$ and $\theta_n(a_j,a_{j+1})$ for $j=1,\dots 2^n$. To find them, we will choose, for each $i>1$, their respective E_i-classes $e_1^i,\dots e_{2^n+1}^i$.
Fix a conjunction $\sigma(x,y)$ in $\rho_n(x,y)$ satisfied by a^*,b^* (we may assume $\sigma(x,y) \vdash x \neq y$).
Let $1<i\leq N$. If the term $E_i(x,y)$ appears in $\sigma_n(x,y)$, let $e_1^i=\dots=e_{2^n+1}^i=[a']_{E_i}=[b']_{E_i}$. If not, since $I \leq 2^n$, choose different classes $e_1^i=[a']_{E_i}, e_2^i,\dots,e_{2^n+1}^i=[b']_{E_i}$ such that $R_i(x,y)$ holds if the classes of x,y are e_j^i, e_{j+1}^i for any $1\leq j<2^n+1$. This is because inside any of the circles $C_2,\dots,C_N$, given two points p,q, we can always find points $p_1=p, p_2,\dots,p_{2^n+1}=q$ such that $d(p_j,p_{j+1})<1/2N$ for $j=1,\dots,2^n$, and $1/2N \leq 1/2i$.
Let $i>N$. Choose j minimal such that $T(x,y) \vdash R_i^j(x,y)$ appears in $T(x,y)$ (Notice here that the language of $\theta_n(x,y)$ can be bigger than the language of $\theta_0(x,y)$). Again, if the term $E_i(x,y)$ appears in $\sigma(x,y)$, let $e_1^i=\dots=e_{2^n+1}^i=[a']_{E_i}=[b']_{E_i}$. If not,

since $R_i^j(a',b')$, choose different classes $e_1^i = [a']_{E_i}, e_2^i, \ldots, e_{2^n+1}^i = [b']_{E_i}$ such that $R_i^j(x,y)$ holds if the classes of x,y are e_j^i, e_{j+1}^i for any $1 \leq j < 2^n+1$.
This completes the choice of the classes. Now let $a' = a_1, a_2 \ldots, a_{2^n+1} = b'$ be elements of C^* such that $[a_j]_{E_i} = e_j^i$ for all $i > 1$ and $1 \leq j \leq 2^n$. By the choice of the classes, it is easy to see that $T(a_j, a_{j+1})$ holds for all $1 \leq j \leq 2^n$. This implies that $P_n(a_j, a_{j+1})$ and therefore $S_n(a_j, a_{j+1})$ holds for all $1 \leq j \leq 2^n+1$, as we wanted. $\diamond$

$\square$

Lemma 3.1.11. *Let $a_i, b_i \in C_i$ for all $i > 1$. Then $(a_i)_{i>1} \overset{\text{L}}{\equiv} (b_i)_{i>1}$ in C^* if and only if $(a_i)_{i>1} \overset{\text{L}}{\equiv} (b_i)_{i>1}$ in $\bigsqcup_{i>1} C_i$.*

Proof. $\Rightarrow$) Suppose $(a_i)_{i>1} \overset{\text{L}}{\equiv} (b_i)_{i>1}$ in C^*. Without loss of generality, we may assume there is $C' \prec C^*$ and $f \in \text{Aut}(C^*/C')$ such that $f((a_i)_{i>1}) = (b_i)_{i>1}$. Let π_i be the projection map from C^* to C_i (remember that no new elements where added to the C_i's when going from C to C^*). For all $i > 1$, $\pi_i(C') \subseteq C_i$. Moreover, by elimination of quantifiers, $\pi_i(C') \prec C_i$ for all $i > 1$.
Now let $f_i : C_i \to C_i$ be defined as follows: for $c_i \in C_i$, let $f_i(c_i) = \pi_i(f(c))$ for any $c \in \pi_i^{-1}(c_i)$ [1]. Let f' be the union of the f_i's. Clearly

$$f' \in \text{Aut}\left(\bigsqcup_{i>1} C_i / \bigsqcup_{i>1} \pi_i(C')\right),$$

$\bigsqcup_{i>1} \pi_i(C') \prec \bigsqcup_{i>1} C_i$, and $f'(a_i) = b_i$ for all $i > 1$, thus $(a_i)_{i>1} \overset{\text{L}}{\equiv} (b_i)_{i>1}$ in $\bigsqcup_{i>1} C_i$.
$\Leftarrow$) Without loss of generality, we may assume there are $C_i' \prec C_i$ and $f_i \in \text{Aut}(C_i/C_i')$ such that $f_i(a_i) = b_i$ for all $i > 1$. Let $f \in \text{Aut}\left(\bigsqcup_{n>1} C_n\right)$ be the union of the f_i's. By remark 3.1.9 (2.), there is $f^* \in \text{Aut}(C^*)$ whose canonical projection to $\text{Aut}\left(\bigsqcup_{n>1} C_n\right)$ is f. Observe that $\prod_{n>1} C_n' \prec C^*$, f^* fixes pointwise $\prod_{n>1} C_n'$, and $f^*((a_i)_{i>1}) = (b_i)_{i>1}$, showing that $(a_i)_{i>1}$ and $(b_i)_{i>1}$ have the same Lascar strong type in C^*. $\square$

Proposition 3.1.12. *Let $a_i, b_i \in C_i$ for all $i > 1$. Then $(a_i)_{i>1} \overset{\text{L}}{\equiv} (b_i)_{i>1}$ in C^* if and only if there is $n < \omega$ such that $d(a_i, b_i) \leq n$ for all $i > 1$.*

Proof. $\Rightarrow$) Observe that there is $n < \omega$ such that $d(a_i, b_i) \leq n$ for all $i > 1$ if and only if there is a Lascar strong automorphism of $\bigsqcup_{i>1} C_i$ sending the (infinite) tuple $(a_i)_{i>1}$ to $(b_i)_{i>1}$, i.e., if and only if $(a_i)_{i>1} \overset{\text{L}}{\equiv} (b_i)_{i>1}$ in $\bigsqcup_{i>1} C_i$. By the previous lemma, that happens if and only if $(a_i)_{i>1} \overset{\text{L}}{\equiv} (b_i)_{i>1}$ in C^*. $\square$

[1] If $c, c' \in \pi^{-1}(a_i)$, then $E_i(c, c')$, and since $f \in \text{Aut}(\mathfrak{C})$, it is true also that $E_i(f(c), f(c'))$, and therefore $\pi_i(f(c)) = \pi_i(f(c'))$.

Corollary 3.1.13. *Neither* $\mathrm{Th}(\bigsqcup_{n>1} C_n)$ *nor* $\mathrm{Th}(C^*)$ *are* G*-compact.*

Proof. Consider, for each $n > 1$, two diametrically opposed points $a_n, a'_n \in C_n$, and a Lascar strong automorphism $f_n \in \mathrm{Aut}(C_n)$ such that $f_n(a_n) = a'_n$.

Let $f \in \mathrm{Aut}(\bigsqcup_{n>1} C_n)$ be the union of the f_n's. Note that f is a limit point of $\mathrm{Autf}(\bigsqcup_{n>1} C_n)$ (since f agrees with the product of finitely many f_n's, which is strong) but it's not Lascar strong, since there is no finite bound for the distance between the (infinite) tuples $(a_n : n < \omega)$ and $(f(a_n) : n < \omega)$. By fact 1.1.2 (3.), $\mathrm{Th}(\bigsqcup_{n>1} C_n)$ is not G-compact.

For the second case, let $c, c' \in C^*$ be such that $c(n) = a_n$ and $c'(n) = a'_n$ for each $n > 1$. By proposition 3.1.10, $c \overset{\mathrm{KP}}{\equiv} c'$, but we know, by proposition 3.1.12, that $c \overset{\mathrm{L}}{\not\equiv} c'$, showing that $\overset{\mathrm{L}}{\equiv}$ is not type-definable for finite tuples. By fact 1.1.2 (2.), $\mathrm{Th}(C^*)$ is not G-compact. $\square$

3.2 G-compactness of T does not imply G-compactness of T_A

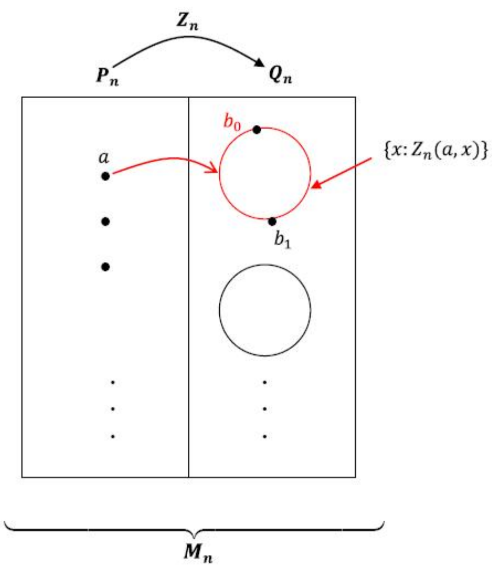
The next examples came up as a result of several discussions with Prof. Ludomir Newelski during my visit at the Mathematical Institute of the Wroclaw University in March 2006 and previous conversations with Prof. Enrique Casanovas at the University of Barcelona.

Type-definability of $\overset{\mathrm{L}}{\equiv}_A$ for finite tuples is preserved

Example 1. For each $n > 1$, let M_n be a saturated model of the theory T_n in the language $\mathcal{L}_n = \{P_n, Q_n, Z_n, B_n, L_n, R_n^m, S_n^m : 1 \leq m \leq n\}$, where:

1. P_n and Q_n are infinite disjoint unary predicates.
2. Z_n is a binary relation symbol on $P_n \times Q_n$.
3. For any two different elements $x, y \in P_n$, $\{z : Z_n(x, z)\} \cap \{z : Z_n(y, z)\} = \emptyset$.
4. For each $x \in P_n$, the set $\{z : Z_n(x, z)\}$ is non-empty and is equipped with the same structure as the circle C_n described before in the language $\{B_n, L_n, R_n^m, S_n^m : 1 \leq m \leq n\}$.
5. For each $y \in Q_n$ there is $x \in P_n$ such that $Z_n(x, y)$.

The following figure illustrates what M_n looks like.



Proposition 3.2.1. *1. For each $n > 1$ and tuples x, y (possibly infinite) of M_n (of the same length) such that $x \equiv y$, $d_\emptyset(x, y) \leq 2$.*

2. Fix $1 < n < \omega$ and let $a \in P_n(M_n)$. There are elements x, y of M_n such that $x \stackrel{\mathrm{L}}{\equiv}_a y$ but $d_a(x, y) \geq n$.

Proof. 1. Due to the existence of infinitely many circles in M_n, we can easily find an automorphism of M_n sending x to y and fixing an elementary submodel of M_n. Just observe that any subset of M_n with infinitely many circles and their respective points in P_n is an elementary substructure of M_n. By lemma 1.1.3, $d_\emptyset(x, y) \leq 2$.
2. Introducing an element $a \in P_n(M_n)$ to the language leads us to the case of just one circle where the diameter is $\geq n$, since any elementary substructure of (M_n, a) should include a and an elementary substructure of the circle $\{y : Z_n(a, y)\}$. Take two diametrically opposed elements b_0, b_1 such that $Z_n(a, b_0)$ and $Z_n(a, b_1)$ and observe that $b_0 \stackrel{\mathrm{L}}{\equiv}_a b_1$ and $d_a(b_0, b_1) = n$.

□

Let $M = \bigsqcup_{n>1} M_n$ be the coproduct, or disjoint sum, of the M_n's. It's just the family of structures $(M_n : n > 1)$ considered as a many-sorted structure. The n-th sort corresponds to M_n equipped with all its $\mathcal{L}_n$-structure. No additional relation or function symbols.

Corollary 3.2.2. *Let $A = \{a_n : n > 1\} \subseteq M$ such that for every $n > 1$, $a_n \in P_n(M_n)$. Then $\mathrm{Th}(M)$ is G-compact over $\emptyset$ but it is not G-compact over A (T_A is not G-compact over $\emptyset$).*

Proof. By the previous proposition we know that the $\emptyset$-diameter of the Lascar strong types over $\emptyset$ in M is 2, but there is not a bound for the A-diameter of the Lascar strong types over A. By theorem 1.1.4, we have what we want. We also can show as in corollary 3.1.13 that $\mathrm{Autf}(M/A)$ is not closed in $\mathrm{Aut}(M/A)$. □

In the former example we used an infinite set of parameters in order to lose G-compactness (over $\emptyset$) going from T to T_A. Using the same ideas we now show that G-compactness (over $\emptyset$) can be displaced just by adding one parameter to the language.

Example 2. Let M be a saturated model of the theory T in the language $\mathcal{L} = \{P, Q\} \cup \{Z_n, B_n, L_n, R_n^m, S_n^m : 1 \leq m \leq n, n > 1\}$ given by the following axioms.

1. P and Q are infinite disjoint unary predicates.

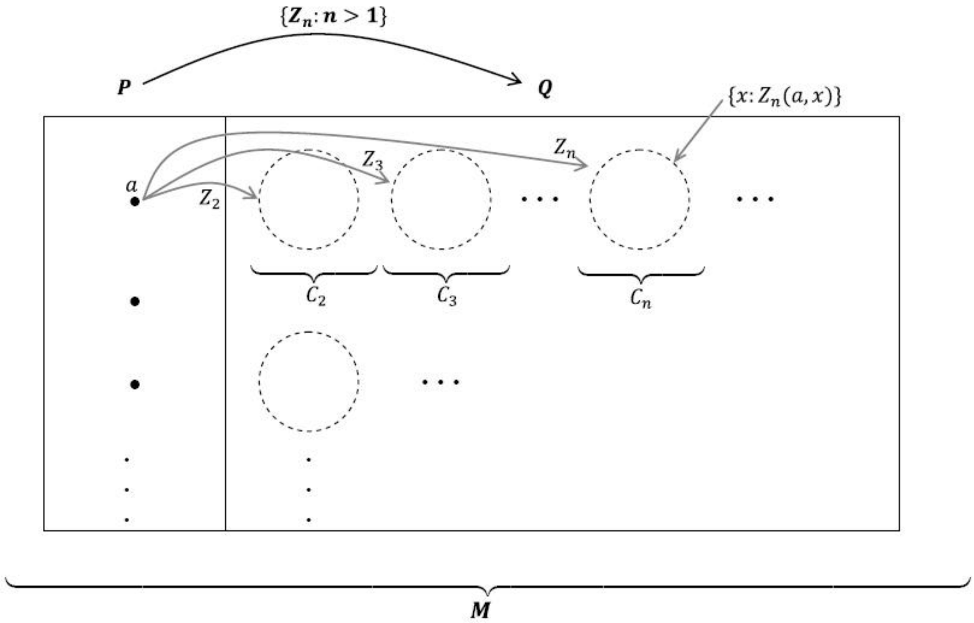
2. For every $n > 1$, Z_n is a binary relation symbol on $P \times Q$.

3. $\{z : Z_i(x, z)\} \cap \{z : Z_j(y, z)\} = \emptyset$ whenever $(x, i), (y, j)$ are different tuples of $P \times \omega^*$.

4. For each $x \in P$ and each $n > 1$, the set $\{z : Z_n(x, z)\}$ is non-empty and is equipped with the same structure as the circle C_n described before in the language $\{B_n, L_n, R_n^m, S_n^m : 1 \leq m \leq n\}$.

5. For each $y \in Q$ there is $x \in P$ and $n > 1$ such that $Z_n(x, y)$.

The following figure illustrates what M looks like, except for an infinite set of points which would appear in Q and would not belong to any of the circles.



Proposition 3.2.3. *1. For each $n > 1$ and tuples x, y (possibly infinite) of M (of the same length) such that $x \equiv y$, $d_\emptyset(x, y) \leq 2$. Hence T is G-compact over $\emptyset$.*

2. Let $a \in P(M)$. For each $n > 1$, there are elements x, y of M such that $x \overset{\text{L}}{\equiv}_a y$ but $d_a(x, y) \geq n$. Hence T is not G-compact over $\{a\}$.

Proof. 1. As in proposition 3.2.1. If $x \equiv y$, there is $M' \prec M$ such that $x \equiv_{M'} y$, hence $d_\emptyset(x, y) \leq 2$. This is, again, due to the existence of infinitely many points in P and circles in Q. By fact 1.1.4, T is G-compact.

2. Fix $n > 1$. Take two diametrically opposed elements b_0, b_1 such that $Z_n(a, b_0)$ and $Z_n(a, b_1)$ and observe that $d_a(b_0, b_1) = n$. This shows that there is no

bound on the a-diameters of Lascar strong types over $\{a\}$. By fact 1.1.4, T is not G-compact over $\{a\}$.

$\square$

Type-definability of $\overset{\text{L}}{\equiv}_A$ for finite tuples is displaced

In the previous examples type-definability of $\overset{\text{L}}{\equiv}$ for finite tuples was preserved from T to T_A; in both cases $\overset{\text{L}}{\equiv}_A = \overset{\text{KP}}{\equiv}_A$ for finite tuples. In the next example we will see the case where type-definability of $\overset{\text{L}}{\equiv}$ is displaced because of finite tuples. Namely, we will give a G-compact (over $\emptyset$) theory T and find elements a, b, b' such that $b \overset{\text{KP}}{\equiv}_a b'$ but $d_a(b, b') = \infty$, i.e. $b \overset{\text{L}}{\not\equiv}_a b'$.

Example 3. For each $n > 1$, let $\mathcal{L}_n = \{P_n, Q_n, Z_n, B_n, L_n, R_n^m, S_n^m : 1 \leq m \leq n\}$ and M_n be as in *Example 1* above. Now, let $M = \prod_{n>1} M_n$ be the product of the M_n's. Analogous to the construction of $\prod_{n>1} C_n$ described in section 3.1, an element of M is a mapping f defined on $\omega \setminus \{0, 1\}$ and such that $f(n) \in M_n$ for every $n > 1$. Its language is $\mathcal{L} = \bigcup_{n>1} \mathcal{L}'_n$, where $\mathcal{L}'_n = \mathcal{L}_n \cup \{E_n\}$, and we interpret it as follows. For any $f, f' \in M$, $n > 1, m \leq n$,

- $P_n(f)$ holds if and only if $P_n(f(n))$ holds in M_n (the same for Q_n).
- $Z_n(f, f')$ holds if and only if $P_n(f)$, $Q_n(f)$ and $Z_n(f(n), f'(n))$ holds in M_n.
- $R_n^m(f, f')$ holds if and only if $Q_n(f), Q_n(f')$, there is $g \in M$ such that $P_n(g)$ and $Z_n(g, f) \wedge Z_n(g, f')$ (i.e., their n-th projections lie in the same circle) and $R_n^m(f(n), f'(n))$ holds in M_n. Similarly for B_n, L_n and S_n^m.
- E_n is a binary relation symbol interpreted as follows:

  $$E_n(f, f') \Leftrightarrow f(n) = f'(n).$$

  E_n is an equivalence relation and we can recover the structure M_n with all its $\mathcal{L}_n$-structure taking the quotient M/E_n.

Again, let $E = \bigcap_{n>1} E_n$ and observe that for every $f \in M$, $M/E = \{f\}$. Adjoining a suitable large number of new elements to each E-class one gets a saturated elementary extension M^* of M. Note that in passing to M^* no new elements were added to any of the M_n's. With a back-and-forth argument it is easy to see that $\text{Th}(M^*)$ admits elimination of quantifiers in the language $\mathcal{L}$.

Proposition 3.2.4. $\text{Th}(M^*)$ *is G-compact.*

Proof. As in propositions 3.2.1 and 3.2.3, if x, y are two tuples (possibly infinite) of M^* with the same quantifier-free type, then we can find an automorphism of M^* sending x to y and fixing an elementary substructure. By lemma 1.1.3, 2 is a bound for the diameter of Lascar strong types, and by theorem 1.1.4 $Th(M)$ is G-compact. $\square$

Now fix $a \in M$ such that $P_n(a)$ holds for all $n > 1$ and elements $b, b' \in M$ such that for all $n > 1$, $Q_n(b), Q_n(b'), Z_n(a, b), Z_n(a, b')$ and $b(n)$ and $b'(n)$ are diametrically opposed.

Proposition 3.2.5. *$b \overset{\text{KP}}{\equiv}_a b'$ in M^*.*

Proof. We follow the proof of proposition 3.1.10. By definition, $b \overset{\text{KP}}{\equiv}_a b'$ if and only if b and b' are equivalent under any $\{a\}$-type-definable bounded equivalence relation, and we know that any such relation can be written as the intersection of a family $\{\theta_n(x, y) : n < \omega\}$ of thick formulas of $\mathcal{L}_a$ such that $\theta^2_{n+1} \subseteq \theta_n$ for every $n < \omega$, i.e., $\theta_{n+1}(x, y) \wedge \theta_{n+1}(y, z) \vdash \theta_n(x, z)$.
Assume, searching for a contradiction, that $\neg\theta_0(b, b')$ for a formula $\theta_0(x, y)$ of some such family of thick formulas $\{\theta_n(x, y) : n < \omega\}$ consistent with $\text{tp}(bb'/a)$. Let $\mathcal{L}'_2 \cup \ldots \mathcal{L}'_N \cup \{a\}$ be the language of θ_0 and choose n such that $N < 2^n$. Observe that

$$\theta_n^{2^n} \subseteq \theta_{n-1}^{2^{n-1}} \cdots \subseteq \theta_2^4 \subseteq \theta_1^2 \subseteq \theta_0.$$

By quantifier elimination, the formula $\theta_n(x, y)$ is of the form

$$\theta_n(x, y) = \bigvee_{i=1}^{l} \sigma_i(x, y)$$

for some $l < \omega$, where each σ_i is a conjunction of some of the relations

$$\{=, P_k, Q_k, Z_k, L_k, R^m_k, S^m_k, E_k : 1 < k \leq N, 1 \leq m \leq k\}$$

and their negations. We can also assume that for each $j = 1, \ldots l$,

$$\sigma_j \vdash \bigwedge_{i=2}^{N} (Z_i(a, x) \wedge Z_i(a, y)),$$

meaning that for any $i = 2, \ldots, N$ and $j = 1, \ldots, l$, the i-th coordinates of a pair satisfying $\sigma_j(x, y)$ lie in circle attached to $a(i)$. This is because the families of thick formulas we're interested in are consistent with $\text{tp}(bb'/a)$.

Claim 1. At least one of the σ_i's is not negative in any of the relations R^m_k $(1 < k \leq N, 1 \leq m \leq k)$. (This means that at least in one conjunction there is no lower bound for the distance between any two comparable coordinates of x and y).

Proof. Otherwise, in each σ_i there is a term $\neg R_{k_i}^{m_i}$. But the relation

$$\bigwedge_{i=1}^{l} R_{k_i}^{m_i} \vee \bigvee_{i=2}^{N} \neg(Z_i(a,x) \wedge Z_i(a,y))$$

is thick, since it is impossible to find an infinite antichain for it, i.e., infinitely many elements such that their k_i-th projections are in the circle attached to a and at distance $> m_i/k_i$. Since

$$\theta_n(x,y) = \bigvee_{i=1}^{l} \sigma_i \vdash \bigvee_{i=1}^{l} \neg R_{k_i}^{m_i} \wedge \bigwedge_{i=2}^{N} (Z_i(a,x) \wedge Z_i(a,y)),$$

then

$$\bigwedge_{i=1}^{l} R_{k_i}^{m_i} \vee \bigvee_{i=2}^{N} \neg(Z_i(a,x) \wedge Z_i(a,y)) \vdash \neg\theta_n(x,y).$$

This implies that $\neg\theta_n(x,y)$ is thick and $\theta_n(x,y)$ is not, which is a contradiction. $\diamond$

We write $\theta_n(x,y) = \rho_n(x,y) \vee \nu_n(x,y)$, where $\rho_n(x,y)$ is the disjunction of the conjunctions which are not negative in any of the relations R_k^m, and $\nu_n(x,y)$ is the disjunction of the remaining ones.

Claim 2. $\rho_n(x,y) \nvdash \bigvee_{i=2}^{N} E_i(x,y) \vee x = y$.

Proof. Otherwise, choose in each conjunction of $\nu_n(x,y)$ a relation $R_{k_i}^{m_i}$ appearing negatively (say for $i = 1, \dots l'$) and observe that

$$\bigwedge_{i=2}^{N} \neg E_i(x,y) \wedge x \neq y \wedge \bigwedge_{i=1}^{l'} R_{k_i}^{m_i} \vdash \neg\rho_n(x,y) \wedge \neg\nu_n(x,y) \vdash \neg\theta_n(x,y).$$

This implies that we can find an infinite sequence $(a_i : i < \omega)$ of different elements in M^* such that $\neg\theta_n(a_i, a_j)$, whenever $i < j$. Therefore, $\theta_n(x,y)$ is not thick, and we have again a contradiction. $\diamond$

Since $\rho_n(x,y) \wedge \bigwedge_{i=2}^{N} \neg E_i(x,y) \wedge x \neq y$ is consistent, we can find elements $c, c' \in M^*$ such that $\rho_n(c,c')$ and such that $E_i(c,c')$ if and only if $E_i(b,b')$ for $i = 2, \dots, N$. We can also find distinct elements d, d' such that

- $E_i(d,b)$ and $E_i(d',b')$ for $i = 2, \dots N$.
- $E_i(d,c)$ and $E_i(d',c')$ for $i > N$.

Claim 3. $\theta_n^{2^n}(d,d')$.

If the claim is true, then $\theta_0(d,d')$. By the choice of the E_i-classes and elimination of quantifiers, the pair d, d' satisfies the same formulas of $\mathcal{L}_1 \cup \cdots \cup \mathcal{L}_n$ than the pair b, b'. This implies that $\theta_0(b,b')$, which gives us the desired contradiction.

Proof Claim 3. It is enough to find $d_1, \dots, d_{2^n+1}$ such that $d_1 = d$, $d_{2^n+1} = d'$ and $\theta_n(d_j, d_{j+1})$ for $j = 1, \dots 2^n$. To find them, we will choose, for each $i > 1$, their respective E_i-classes $e^i_1, \dots e^i_{2^n+1}$.
Fix a conjunction $\sigma(x, y)$ in $\rho_n(x, y)$ satisfied by c, c' (we may assume $\sigma(x, y) \vdash x \neq y$).
Let $1 < i \leq N$. Since $I \leq 2^n$, choose different classes $e^i_1 = [d]_{E_i}, e^i_2, \dots, e^i_{2^n+1} = [d']_{E_i}$ such that $R_i(x, y)$ holds if the classes of x, y are e^i_j, e^i_{j+1} for any $1 \leq j < 2^n + 1$. This is because inside any of the circles $C_2, \dots, C_N$, given two points p, q, we can always find points $p_1 = p, p_2, \dots, p_{2^n+1} = q$ such that $d(p_j, p_{j+1}) < 1/2N$ for $j = 1, \dots, 2^n$, and $1/2N \leq 1/2i$.
Let $i > N$. Choose j minimal such that $\sigma(x, y) \vdash R^j_i(x, y)$ appears in $\sigma(x, y)$ (Notice here that the language of $\theta_n(x, y)$ can be bigger than the language of $\theta_0(x, y)$). Since $R^j_i(d, d')$ (because for $i > N$, $E_i(c, c)$ and $E_i(c', d')$), choose different classes $e^i_1 = [d]_{E_i}, e^i_2, \dots, e^i_{2^n+1} = [d']_{E_i}$ such that $R^j_i(x, y)$ holds if the classes of x, y are e^i_j, e^i_{j+1} for any $1 \leq j < 2^n + 1$.
This completes the choice of the classes. Now let $d = d_1, d_2 \dots, d_{2^n+1} = d'$ be elements of M^* such that $[d_j]_{E_i} = e^i_j$ for all $i > 1$ and $1 \leq j \leq 2^n$. By the choice of the classes, it is easy to see that $\sigma(d_j, d_{j+1})$ holds for all $1 \leq j \leq 2^n$. This implies that $\rho_n(d_j, d_{j+1})$ and therefore $\theta_n(d_j, d_{j+1})$ holds for all $1 \leq j \leq 2^n + 1$, as we wanted. $\diamond$

$\square$

Proposition 3.2.6. *Let $b, b' \in M$ be elements of M^* such that for all $n > 1$, $Q_n(b), Q_n(b'), Z_n(a, b), Z_n(a, b')$ and $b(n)$ and $b'(n)$ are diametrically opposed. Then $d_a(b, b') = \infty$.*

Proof. Suppose $d_a(b, b') = n$ for some $n < \omega$. Then $b \overset{\mathrm{L}}{\equiv}_a b'$ and we can find elementary substructures $N_1, \dots N_n \prec M^*$ containing a, and elements $b_1, b_2, \dots, b_{n+1} \in M$ such that $b_1 = b, b_{n+1} = b'$ and $b_i \equiv_{N_i} b_{i+1}$ for $i = 1, \dots, n$. Observe that for all $j > 1$ and $i \in \{1, \dots, n\}$, $b_i(j) \equiv_{\pi_j(N_i)} b_{i+1}(j)$, where π_j is the projection map from M^* to M_j (remember that no new elements where added to the M_i's when going from M to M^*). As in the proof of proposition 3.1.11, $\pi_j(N_i) \prec M_j$ and $a(j) \in \pi_j(N_i)$ for all $j > 1$ and $i \in \{1, \dots, n\}$.
Now fix $k > n$. Since $b_i(k) \equiv_{\pi_k(N_i)} b_{i+1}(k)$ for all $i \in \{1, \dots, n\}$, in particular we know that $b_i(k) \equiv_{\Sigma^k_i} b_{i+1}(k)$ where $\Sigma^k_i = \{x \in \pi_k(N_i) : Z_k(a(k), x)\}$. Since Σ^k_i is isomorphic to an elementary substructure of the circle C_k, by proposition 3.1.7, we can conclude that $S_k(b_i(k), b_{i+1}(k))$ holds for all $i \in \{1, \dots, n\}$. Since $b_1(k)$ and $b_{n+1}(k)$ are diametrically opposed, this is impossible. $\square$

Corollary 3.2.7. *$Th(M^*)$ is not G-compact over $\{a\}$*

Proof. By the previous two propositions, $\stackrel{\text{L}}{\equiv}_a \neq \stackrel{\text{KP}}{\equiv}_a$ for finite tuples. By fact 1.1.2, $Th(M^*)$ is not G-compact over $\{a\}$. $\square$

3.3 A new proof for the finite diameter of type-definable Lascar strong types

The notions of *c-free* and *weakly c-free* extensions over a complete type were introduced in [17]. Using these tools, Newelski proves strong results on countable coverings of groups and types. In this section, we slightly generalize these definitions to *c-free* and *weakly c-free* extensions over a *partial type* with special properties, in order to give a more direct proof of the fact that type-definable Lascar strong types have a finite diameter (Corollary 1.8, [16]). This was proved before in [16] by way of contradiction using the so called open analysis, and during my visit at the Mathematical Institute of the Wroclaw University, Prof. Newelski suggested that this could be also proved using these new techniques.

Throughout this section we will assume that $\pi(x)$ is a partial type over A such that for every $a, b \models \pi(x)$, there is an automorphism $f \in \mathrm{Aut}(\mathfrak{C})$ such that $f(a) = b$ and $\pi(x) \equiv \pi^f(x)$ [2].

Definition 3.3.1. *A set $U \subseteq \mathfrak{C}$ is* c-free *over π if there are $n < \omega$ and automorphisms $f_0, \ldots, f_{n-1} \in \mathrm{Aut}(\mathfrak{C})$ such that*

$$\begin{aligned} i) \quad & \pi(\mathfrak{C}) \subseteq \bigcup_{i<n} f_i(U) \\ ii) \quad & \pi \equiv \pi^{f_i} \text{ for every } i < n \end{aligned}$$

A formula φ is c-free over π if $\varphi(\mathfrak{C})$ is c-free over π. A type $q(x)$ is c-free over π if every formula $\varphi(x)$ such that $q(x) \vdash \varphi(x)$ is c-free over π.

Definition 3.3.2. *A set $U \subseteq \mathfrak{C}$ is* weakly c-free *over π if for some $V \subseteq \mathfrak{C}$ which is not c-free over π, $U \cup V$ is c-free over π. A formula $\varphi(x)$ is weakly c-free over π if the set $\varphi(\mathfrak{C})$ is. A type $q(x)$ is weakly c-free over π if every formula $\varphi(x)$ such that $q(x) \vdash \varphi(x)$ is weakly c-free over π.*

The following are general properties of weakly c-free sets, types and formulas.

Lemma 3.3.3. *Assume U is a definable subset of $\mathfrak{C}$. Then the following conditions are equivalent.*

1. *U is weakly c-free over π.*
2. *$\bigcap_{i<n} f_i(U)^c$ is not c-free over π, for some $f_0, \ldots f_{n-1} \in \mathrm{Aut}(\mathfrak{C})$ such that $\pi^{f_i} \equiv \pi$ for all $i < n$.*

[2]Note that when $\pi(x)$ is a complete type over A we get a particular case of our more general context. We can actually find $f \in \mathrm{Aut}(\mathfrak{C}/A)$. Note also that all the the elements of $\pi(x)$ have the same type over $\emptyset$.

3. *For some definable set $V \subseteq \mathfrak{C}$ that is not c-free over π, $U \cup V$ is c-free over π.*

Proof. 1. $\to$ 2. If U is weakly c-free over π, then there is $V \subseteq \mathfrak{C}$ that is not c-free over π such that $U \cup V$ is c-free over π. Thus, there are $f_0, \dots, f_{n-1} \in \mathrm{Aut}(\mathfrak{C})$ such that $\pi^{f_i} \equiv \pi$ for all $i < n$ and

$$\begin{aligned}
\pi(\mathfrak{C}) &\subseteq \bigcup_{i<n} f_i(U \cup V) \\
\Rightarrow \pi(\mathfrak{C}) &\subseteq \bigcup_{i<n} f_i(U) \cup \bigcup_{i<n} f_i(V) \\
\Rightarrow \left(\bigcap_{i<n} f_i(U)^c\right) \cap \pi(\mathfrak{C}) &\subseteq \bigcup_{i<n} f_i(V).
\end{aligned}$$

Since V is not c-free over π, neither is $\bigcup_{i<n} f_i(V)$. Then $\left(\bigcap_{i<n} f_i(U)^c\right) \cap \pi(\mathfrak{C})$ is also not c-free over π, which implies that $\bigcap_{i<n} f_i(U)^c$ is not c-free over π.

2. $\to$ 3. Let $V = \bigcap_{i<n} f_i(U)^c$. V is definable and non-c-free over π. Let $f_n = id$, and observe that

$$\begin{aligned}
\bigcup_{i \leq n} f_i(U \cup V) &= \bigcup_{i<n} f_i(U \cup V) \cup (U \cup V) \\
&= \bigcup_{i<n} f_i(U \cup V) \cup \left(U \cup \left(\bigcap_{i<n} f_i(U)^c\right)\right) \\
&\supseteq \left(\bigcap_{i<n} f_i(U)^c\right)^c \cup \left(\bigcap_{i<n} f_i(U)^c\right) \\
&\supseteq \pi(\mathfrak{C}),
\end{aligned}$$

showing that $U \cup V$ is c-free over π.

3. $\to$ 1. is clear. $\square$

Lemma 3.3.4. 1. *If U_1, U_2 are not weakly c-free over π, then $U_1 \cup U_2$ is not weakly c-free over π.*

2. *If $q(x)$ is a (partial) type over $B \supseteq A$ that is weakly c-free over π, then some $q'(x) \in S(B)$ extending $q(x)$ is weakly c-free over π. Necessarily, $\pi(x) \subseteq q'(x)$.*

Proof. 1. Let $V \subseteq \mathfrak{C}$ be non-c-free over π. Since U_2 is not weakly c-free over π, $U_2 \cup V$ is not c-free over π. And since U_1 is not weakly c-free over π, then $U_1 \cup U_2 \cup V$ is also not c-free over π. Thus, $U_1 \cup U_2$ is not weakly c-free over π.

2. By the previous point, if $q(x)$ is weakly c-free over π, then for every $\varphi \in L(B)$, either $q(x) \cup \{\varphi(x)\}$ or $q(x) \cup \{\neg\varphi(x)\}$ are weakly c-free over π. Clearly $\pi(x) \subseteq q'(x)$. $\square$

Consider the following sets of types:

1. $P = \{q(x,y) \in S_{xy}(\emptyset) : q(x,y) \cup \pi(x) \cup \pi(y)$ is consistent$\}$.

2. $P_{wcf} = \{q(x,y) \in P : q(a,y)$ is weakly c-free over π for some (all) $a \models \pi\}$.

Remark 3.3.5. *P and P_{wcf} are closed and nonempty.*

Proof. For P is clear. Let $a \models \pi$. Since $\pi(y)$ is weakly c-free over π, it can be extended to a complete type $q(a,y)$ which is weakly c-free over π. Now let $\Sigma = \{\varphi(y) \in L(a) : \varphi$ is not weakly c-free over $\pi\}$, and let $\Gamma = \{\varphi(x,y) \in L : \varphi(a,y) \in \Sigma\}$. Then

$$P_{wcf} = \{q(x,y) \in P : q(x,y) \supseteq \{\neg\varphi(x,y) : \varphi(x,y) \in \Gamma\}\}$$

□

Proposition 3.3.6. *Assume $S \subseteq P_{wcf}$ is non-empty and relatively open. Then there are finitely many $c_i \models \pi$, $1 \leq i \leq k$, such that for every $b \models \pi$ there is $d \models \pi$ such that*

1. $\mathrm{tp}(b,d) \in S$.

2. $\mathrm{tp}(c_i,d) \in S$ *for some* $1 \leq i \leq k$.

Proof. Since S is non-empty and relatively open, let $\varphi(x,y) \in L$ be a formula such that $S \supseteq P_{wcf} \cap [\varphi(x,y)] \neq \emptyset$. Since the goal is to find some types inside S, we can assume that $S = P_{wcf} \cap [\varphi(x,y)] \neq \emptyset$.

Fix $c \models \pi$, and observe that $\varphi(c,y)$ is weakly c-free over π.
Let $\psi(e,y)$ be a formula which is not c-free over π such that $\varphi(c,y) \vee \psi(e,y)$ is c-free over π. Let $f_1, \dots f_k \in \mathrm{Aut}(\mathfrak{C})$ such that $\pi^{f_i} \equiv \pi$ for $1 \leq i \leq k$ and

$$\pi(y) \vdash \bigvee_{i=1}^{k} \varphi(c_i,y) \vee \psi(e_i,y) \qquad (\star),$$

where $(c_i,e_i) = (f_i(c), f_i(e))$ $(1 \leq i \leq k)$.

Since $S \subseteq S_{xy}(\emptyset)$ is closed, we can assume $S = \{q(x,y) \in S_{xy}(\emptyset) : q(x,y) \supset \rho(x,y)\}$, for some partial type $\rho(x,y)$ over $\emptyset$.

Claim 1. $(\varphi(c,\mathfrak{C}) \cup \psi(e,\mathfrak{C})) \setminus \rho(c,\mathfrak{C})$ is not c-free over π.
Proof. Assume it is. Then there are $g_1, \dots, g_n \in \mathrm{Aut}(\mathfrak{C})$ fixing $\pi(\mathfrak{C})$ such that

$$\pi(\mathfrak{C}) \subseteq \bigcup_{i=1}^{n} g_i((\varphi(c,\mathfrak{C}) \cup \psi(e,\mathfrak{C})) \setminus \rho(c,\mathfrak{C})),$$

i.e., the set of formulas

$$\pi(y) \wedge \bigwedge_{i=1}^{n} ((\neg\varphi(c'_i, y) \wedge \neg\psi(e'_i, y)) \vee \rho(c'_i, y))$$

where $(c'_i, e'_i) = (g_i(c), g_i(e))$, is not satisfiable. By compactness, there are formulas $\alpha_i(x, y)$ $(1 \leq i \leq n)$ such that $\rho(c, \mathfrak{C}) \subseteq \alpha_i(c, \mathfrak{C}) \subseteq \varphi(c, \mathfrak{C})$ and such that

$$\pi(y) \wedge \bigwedge_{i=1}^{n} ((\neg\varphi(c'_i, y) \wedge \neg\psi(e'_i, y)) \vee \alpha_i(c'_i, y))$$

is not satisfiable. Setting $\alpha(x, y) = \bigwedge_{i=1}^{n} \alpha_i(x, y)$, we have that $\rho(c, \mathfrak{C}) \subseteq \alpha(c, \mathfrak{C}) \subseteq \varphi(c, \mathfrak{C})$ and

$$\pi(y) \wedge \bigwedge_{i=1}^{n} ((\neg\varphi(c'_i, y) \wedge \neg\psi(e'_i, y)) \vee \alpha(c'_i, y))$$

is not satisfiable. Thus,

$$\pi(\mathfrak{C}) \subseteq \bigcup_{i=1}^{n} g_i((\varphi(c, \mathfrak{C}) \cup \psi(e, \mathfrak{C})) \setminus \alpha(c, \mathfrak{C})),$$

i.e., $(\varphi(c, \mathfrak{C}) \cup \psi(e, \mathfrak{C})) \setminus \alpha(c, \mathfrak{C})$ is c-free over π. Since $\psi(e, \mathfrak{C}) \cup (\varphi(c, \mathfrak{C}) \setminus \alpha(c, \mathfrak{C}))$ is larger, it is also c-free over π, which implies that $(\varphi(c, \mathfrak{C}) \setminus \alpha(c, \mathfrak{C}))$ is weakly c-free over π (since $\psi(e, \mathfrak{C})$ was not c-free over π). Thus, $(\varphi(c, y) \wedge \neg\alpha(c, y))$ is a partial weakly c-free type over π.

Since $\rho(c, \mathfrak{C}) \subseteq \alpha(c, \mathfrak{C})$, we know that $[\neg\alpha(c, y)] \cap [\rho(c, y)] = \emptyset$. Using the fact that $[\rho(c, y)] = \{q(c, y) : q(x, y) \in P_{wcf}\} \cap [\varphi(c, y)]$, we would conclude that

$$\begin{aligned}
& [\varphi(c, y) \wedge \neg\alpha(c, y)] \cap \{q(c, y) : q(x, y) \in P_{wcf}\} \\
\subseteq \; & [\varphi(c, y)] \cap [\neg\alpha(c, y)] \cap \{q(c, y) : q(x, y) \in P_{wcf}\} \\
= \; & [\rho(c, y)] \cap [\neg\alpha(c, y)] = \emptyset.
\end{aligned}$$

And this is impossible since, by lemma 3.3.4, $(\varphi(c, y) \wedge \neg\alpha(c, y))$ can be extended to a complete weakly c-free type over π. The claim is proved.

With the following claim we finish the proof of the proposition.

Claim 2. For each $b \models \pi$ there is $d \in \rho(b, \mathfrak{C})$ such that $d \in \bigcup_{i=1}^{k} \rho(c_i, \mathfrak{C})$.

Proof. Suppose this is not true. Then we could find $b \models \pi$ such that for every $d \in \rho(b, \mathfrak{C})$, $d \notin \bigcup_{i=1}^{k} \rho(c_i, \mathfrak{C})$. By $(\star)$, for any such d, $d \in \bigcup_{i=1}^{k} (\varphi(c_i, \mathfrak{C}) \cup \psi(e_i, \mathfrak{C}))$, thus we have that

$$\rho(b, \mathfrak{C}) \subseteq \bigcup_{i=1}^{k} ((\varphi(c_i, \mathfrak{C}) \cup \psi(e_i, \mathfrak{C})) \setminus \rho(c_i, \mathfrak{C})). \quad (\star\star)$$

Let $c_{k+1} = b$ and $e_{k+1} = f^*(e)$, where $f^* \in \mathrm{Aut}(\mathfrak{C})$ such that $\pi^{f^*} \equiv \pi$ and $f^*(c) = b$. In this way we ensure that $\varphi(c_{k+1}, \mathfrak{C}) \cup \psi(e_{k+1}, \mathfrak{C})$ is also c-free over π. Thus,

$$\varphi(c_{k+1}, \mathfrak{C}) \cup \psi(e_{k+1}, \mathfrak{C}) \subseteq \bigcup_{i=1}^{k+1} ((\varphi(c_i, \mathfrak{C}) \cup \psi(e_i, \mathfrak{C})) \setminus \rho(c_i, \mathfrak{C})),$$

because $\varphi(c_{k+1}, \mathfrak{C}) \cup \psi(e_{k+1}, \mathfrak{C}) = ((\varphi(c_{k+1}, \mathfrak{C}) \cup \psi(e_{k+1}, \mathfrak{C})) \setminus \rho(b, \mathfrak{C})) \cup \rho(b, \mathfrak{C})$, and, by $(\star\star)$, $\rho(b, \mathfrak{C}) \subseteq \bigcup_{i=1}^{k} ((\varphi(c_i, \mathfrak{C}) \cup \psi(e_i, \mathfrak{C})) \setminus \rho(c_i, \mathfrak{C}))$. Since $\varphi(c_{k+1}, \mathfrak{C}) \cup \psi(e_{k+1}, \mathfrak{C})$ is c-free over π, so is the set on the right hand side. Thus, $(\varphi(c, \mathfrak{C}) \cup \psi(e, \mathfrak{C})) \setminus \rho(c, \mathfrak{C})$ is also c-free over π, contradicting *Claim 1.*
The second claim is proved and so is the proposition.

□

Corollary 3.3.7. *(Corollary 1.8, [16]) Type-definable Lascar strong types have finite diameter.*

Proof. Let $\pi(x)$ be a partial type defining the Lascar strong type of a over $\emptyset$. Clearly π satisfies the conditions required at the beginning. Let $X_n(x, y) = P \cap [nc^n(x, y)]$. Notice that

$$P = \{q(x, y) \in S_{xy}(\emptyset) : q(x, y) \cup \pi(x) \cup \pi(y) \text{ is consistent}\} = \bigcup_{n \in \omega} X_n.$$

By the Baire Category Theorem, there is $N \in \omega$ such that X_N has non-empty interior in P_{wcf}. By the previous proposition, there are finitely many $c_i \models \pi$, $i < k$ such that for every $a, b \models \pi$, there are $c, d \models \pi$ such that

1. $\mathrm{tp}(a, c), \mathrm{tp}(b, d) \in X_N$.
2. $\mathrm{tp}(c_{i*}, c), \mathrm{tp}(c_{j^*}, d) \in X_N$ for some $i^*, j^* < k$.

Let $M = \min\{n \in \omega : \{\mathrm{tp}(c_i, c_j) : i, j < k\} \subseteq X_n\}$. Then clearly

$$\begin{aligned} d(a, b) &\leq d(a, c) + d(c, c_{i^*}) + d(c_{i^*}, c_{j^*}) + d(c_{j^*}, d) + d(d, b) \\ &\leq N + N + M + N + N = 4N + M \end{aligned}$$

□

4

ω-categoricity

4.1 Introduction

A many-sorted ω-categorical theory with countably many sorts which was not G-compact was presented in [4], where the first examples of such theories where exhibited since Lascar had introduced the notion of G-compactness in [13]. A few years later Ivanov constructed in [10] a one-sorted ω-categorical theory which was not G-compact. Motivated by his construction, we prove the following theorem from which we can derive the existence of such an example more directly.

Theorem 4.1.1. *Let T' be a many-sorted ω-categorical theory with countably many sorts. Then there is a one-sorted ω-categorical theory T^* in which T' is stably embeddable.*

Theorem 4.1.1 actually enables us, given the existence of a many sorted ω-categorical theory with countably many sorts having a property $\mathcal{P}$ which is preserved under stable embeddability, to show the existence of a one-sorted ω-categorical theory having the property $\mathcal{P}$. We will show that non-G-compactness is one of such properties.

Given two complete theories T_1 and T_2 with monster models $\mathfrak{C}_1$ and $\mathfrak{C}_2$ respectively, we say that T_2 is *stably embeddable* in T_1 if $\mathfrak{C}_2$ is isomorphic to the full induced structure of $\mathfrak{C}_1^{eq}$ on a collection of sorts, say Σ, such that $\mathfrak{C}_1^{eq} \restriction \Sigma$ is stably embedded in $(\mathfrak{C}_1, \mathfrak{C}_1^{eq} \restriction \Sigma)$. We refer to [7] for stable embeddability.

In the next section we present an ω-categorical theory T_E admitting quantifier elimination, the scaffolding for our construction. We consider the induced structure of T_E^{eq} on a certain countable collection of sorts which are stably embedded; we interpret the initial ω-categorical theory T' over these sorts; and finally we make an expansion of T_E in which T' is stably embeddable. Thanks to Prof. M. Ziegler for valuable suggestions in the presentation of this material.

The theory T_E is interesting on its own from the Shelah's classification point of view. T_E is not simple and does not have the strict order property. Moreover, T_E does not have the so called SOP_1, putting it in the same place as T^*_{feq}, the first example of a theory with such properties, presented by Shelah in [21] and named like that in [23]. We deal with the classification of T_E in section 4.3.

4.2 Proof of Theorem 4.1.1

Consider the language $\mathcal{L}_E = \{E_n : n < \omega\}$ where, for each $n < \omega$, E_n is a $2n$-ary relation symbol, and let T_0 be the theory saying that, for each $n > 0$, E_n is an equivalence relation on the set of n-tuples which does not depend on the order of the tuples and such that all n-tuples with at least one repeated coordinate lie in one isolated E_n-class. Let K be the class of finite models of T_0.

Remark 4.2.1. *K has the Hereditary property (HP), the Joint Embedding property (JEP), the Amalgamation property (AP) and for each n it has only finitely many isomorphism types of size n.*

Let M be the Fraïssé limit of K and let $T_E = \mathrm{Th}(M)$. T_E is ω-categorical and admits elimination of quantifiers. Moreover, T_E is the model-completion of T_0 and we can see that it is given by the following additional axioms, which are satisfied in M.

For any $n \in \omega$ and partitions P_i of $[\{1, \dots, n+1\}]^i$ $(1 \le i \le n)$, let $\varphi_{n,P_1,\dots,P_n}$ be:

$$\forall x_1, \dots, x_n \left(\left(\bigwedge_{i=1}^{n} E_i \approx_{\restriction [\{x_1,\dots,x_n\}]^i} P_i \right) \to \exists x_{n+1} \left(\bigwedge_{i=1}^{n} E_i \approx_{\restriction [\{x_1,\dots,x_{n+1}\}]^i} P_i \right) \right),$$

where $E_i \approx_{\restriction [\{x_1,\dots,x_n\}]^i} P_i$ means that for every $1 \le j_1 < j_2 < \cdots < j_i \le n$ and every $1 \le j'_1 < j'_2 < \cdots < j'_i \le n$, $E_i(\bar{x}_{\bar{j}}, \bar{x}_{\bar{j}'})$ holds if and only if there is $S \in P_i$ such that $\{j_1, \dots, j_i\}, \{j'_1, \dots, j'_i\} \in S$.

These axioms enable us to extend any partial isomorphism between finite structures of T_0 of size n in any possible way given by the position of the new element with respect to the original domain (of size n) according to $E_1, \dots E_n$.

We call Σ the collection of the imaginary sorts $(M^n/E_n)_{n\in\omega}$. Let e be an imaginary element of $M^{\mathrm{eq}} \restriction \Sigma$, say from the sort M^n/E_n, and a finite set $A \subseteq M$, say of size m, such that $e \notin A^n/E_n$. We say that a tuple $\bar{a} = (a_1, \dots, a_n)$ of M is *generic for e over A* if:

1. $A \cap \{a_1, \dots, a_n\} = \emptyset$.

2. $\pi_{E_n}(\bar{a}) = e$.

3. For each $i \leq m$, if $\bar{b} \in (A \cup \{a_1, \ldots, a_n\})^i \smallsetminus A^i$, then $\bar{b}/E_i \notin A^i/E_i$.

Remark 4.2.2. *For every finite set $A \subseteq M$ and every imaginary element $e \in M^n/E_n$ which is not in A^n/E_n, there is a generic tuple for e over A.*

Proof. Apply the axioms n times and make sure each time the partitions isolate completely the new element, i.e., the classes corresponding to the new possible tuples have just one element. □

Proposition 4.2.3. $\mathrm{Th}(M, M^{\mathrm{eq}} \restriction \Sigma)$ *has quantifier elimination in the language* $\mathcal{L}_E \cup \{\pi_{E_1}, \pi_{E_2}, \ldots\}$.

Proof. We do back-and-forth with the partial isomorphisms between finitely generated (π_{E_i}-closed for every $i > 0$) sets. Observe that these sets are infinite (there are infinitely many imaginary sorts), but all their sorts are finite. Let f be one of such isomorphisms. First we do the following. For each imaginary element $e \in \mathrm{dom}(f)$, say of the n-th imaginary sort, for which there is no real tuple $\bar{a} \in \mathrm{dom}(f)$ such that $\pi_{E_n}(\bar{a}) = e$,

1. Find a real n-tuple $\bar{a} = (a_1, \ldots, a_n)$ which is generic for e over the real part of $\mathrm{dom}(f)$ using the previous remark, and let $\bar{b} = (b_1, \ldots, b_n)$ be a realization of $\mathrm{tp}^f(\bar{a}/\mathrm{dom}(f))$.
2. Add $\{(a_i, b_i) : 1 \leq i \leq n\}$ to the graph of f.
3. For each new real tuple $\bar{d}$, say of length $m > 0$, of (the new) $\mathrm{dom}(f)$, add $(\pi_{E_m}(\bar{d}), \pi_{E_m}(f(\bar{d})))$ to the graph of f.

Now let $a \notin \mathrm{dom}(f)$. In case a is imaginary, say of the n-th imaginary sort, just find a new imaginary $b \notin \mathrm{rng}(f)$ of the n-th imaginary sort, and add (a, b) to the graph of f. In case a is real,

1. Let $\bar{c}, \bar{c}'$ be enumerations of the real part of $\mathrm{dom}(f)$ and $\mathrm{rng}(f)$ respectively such that $\bar{c} \equiv_{\mathcal{L}_E} \bar{c}'$, and use the axioms to find an element b such that $\bar{c}a \equiv_{\mathcal{L}_E} \bar{c}'b$.
2. Add (a, b) to the graph of f.
3. For each new real tuple $\bar{d}$, say of length $m > 0$, of (the new) $\mathrm{dom}(f)$, add $(\pi_{E_m}(\bar{d}), \pi_{E_m}(f(\bar{d})))$ to the graph of f.

It is clear that after extending f in this way, it is again an isomorphism between finitely generated sets. □

We say that a many-sorted structure $M = (M_i)_{i \in I}$ is *trivial* if $\mathrm{Aut}(M) = \prod_{i \in I} \mathrm{Sym}(M_i)$, where $\mathrm{Sym}(M_i)$ is the group of all permutations of the set M_i.

Proposition 4.2.4. *$M^{\mathrm{eq}} \restriction \Sigma$, the full induced structure of $(M, M^{\mathrm{eq}} \restriction \Sigma, (\pi_{E_i})_{i>0})$ over $M^{\mathrm{eq}} \restriction \Sigma$ is trivial.*

Proof. We want to see that any permutation of $M^{\mathrm{eq}} \restriction \Sigma$ is elementary in $(M, M^{\mathrm{eq}} \restriction \Sigma)$. By the previous proposition, it's enough to observe that any such permutation respects the formulas of the form $x = y$ and $x \neq y$, where x, y are variables of the same imaginary sort. $\square$

Proposition 4.2.5. *$M^{\mathrm{eq}} \restriction \Sigma$ is stably embedded in $(M, M^{\mathrm{eq}} \restriction \Sigma)$.*

Proof. We check point number *(2)* in lemma 1.4.1. Let a be a tuple of elements of the home sort. For each $n < \omega$, let

$$A_n = \{\pi_{E_n}(a') : a' \text{ is a finite subset of } a \text{ of length } n\}$$

Let $A = \bigcup_{n<\omega} A_n$. It is clear that $|A| \leq |\hat{T}| + |a|$ and $\mathrm{tp}(a/A) \vdash \mathrm{tp}(a/\Sigma)$. $\square$

We assume now that T' is a many-sorted ω-categorical theory in a relational language [1] $\mathcal{L}' = \{R_i : i \in I\}$ with countably many sorts $(S_n)_{n\in\omega}$. Let $(M^{\mathrm{eq}} \restriction \Sigma)'$ be an expansion of $M^{\mathrm{eq}} \restriction \Sigma$ to a model of T', where the n-th sort of T' corresponds to M^n/E_n.

Proposition 4.2.6. $\mathrm{Th}((M, (M^{\mathrm{eq}} \restriction \Sigma)'))$ *is ω-categorical.*

Proof. Let $(A, (A^{\mathrm{eq}} \restriction \Sigma)'), (B, (B^{\mathrm{eq}} \restriction \Sigma)')$ be two countable models of the theory. By ω-categoricity of T_E and T' we can find isomorphisms $f : A \to B$ and $g : (A^{\mathrm{eq}} \restriction \Sigma)' \to (B^{\mathrm{eq}} \restriction \Sigma)'$. Then $f^{-1}g$ is an automorphism of $A^{\mathrm{eq}} \restriction \Sigma$. Since Σ is stably embedded, there is an automorphism h of $(A, A^{\mathrm{eq}} \restriction \Sigma)$ extending $f^{-1}g$. Observe that fh is an isomorphism between $(A, (A^{\mathrm{eq}} \restriction \Sigma)')$ and $(B, (B^{\mathrm{eq}} \restriction \Sigma)')$. $\square$

Now we make an expansion M^* of M. For each relation symbol $R_i \in \mathcal{L}'$ on the sorts $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_l}$, we add a new relation symbol R^* on $M^{n_1 \cdot n_2 \cdots n_l}$ and we interpret it in the following way:

$$M^* \models R_i^*(\bar{a}_1; \bar{a}_2; \ldots; \bar{a}_l) \iff (M, (M^{\mathrm{eq}} \restriction \Sigma)') \models R_i(\pi_{E_{n_1}}(\bar{a}_1), \pi_{E_{n_2}}(\bar{a}_2), \ldots, \pi_{E_{n_l}}(\bar{a}_l))$$

. M^* is ω-categorical since it is isomorphic to the reduct of a definitional expansion of $(M, (M^{\mathrm{eq}} \restriction \Sigma)')$ to its home sort. With the following proposition we prove theorem 4.1.1.

Proposition 4.2.7. *T' is stably embeddable in* $\mathrm{Th}(M^*)$.

[1] Replacing, if necessary, functions by their graphs and constants by predicates.

Proof. We prove that $M^{*eq} \restriction \Sigma$ is stably embedded in $(M^*, M^{*eq} \restriction \Sigma)$. By proposition 4.2.5, any automorphism of $M^{*eq} \restriction \Sigma$ (which is in fact an automorphism of $(M^{\mathrm{eq}} \restriction \Sigma)'$ and therefore an automorphism of $M^{\mathrm{eq}} \restriction \Sigma$) extends to an automorphism of $(M, (M^{\mathrm{eq}} \restriction \Sigma)')$. Since M^* is isomorphic to the reduct of a definitional expansion of $(M, (M^{\mathrm{eq}} \restriction \Sigma)')$ to its home sort, the automorphism extends to an automorphism of $(M^*, M^{*eq} \restriction \Sigma)$. Finally, observe that M' is isomorphic to $M^{*eq} \restriction \Sigma$. Note that everything definable in $M^{*eq} \restriction \Sigma$ from $(M^*, M^{*eq} \restriction \Sigma)$ is already $\emptyset$-definable in $M^{*eq} \restriction \Sigma$ since it is ω-categorical, where invariance implies $\emptyset$-definability. □

With the last theorem we can now show the existence of a one-sorted ω-categorical non G-compact theory. We first prove that non-G-compactness is preserved under stable embeddability.

Proposition 4.2.8. *Let T and T' be two complete theories such that T' is stably embeddable in T. If T is G-compact, then T' is G-compact.*

Proof. Let $\mathfrak{C}, \mathfrak{C}'$ be the monster models of T and T' respectively. By assumption, $\mathfrak{C}'$ is isomorphic to $\mathfrak{C}^{eq} \restriction \Sigma$, the full induced structure of $\mathfrak{C}^{eq}$ to a certain collection of sorts Σ such that $\mathfrak{C}^{eq} \restriction \Sigma$ is stably embedded in $(\mathfrak{C}, \mathfrak{C}^{eq} \restriction \Sigma)$. By theorem 1.1.4, it is enough to observe that any indiscernible sequence of $\mathfrak{C}^{eq} \restriction \Sigma$ is also an indiscernible sequence of $\mathfrak{C}$, showing that the finite bound for the diameter of the Lascar strong types in T works also as a bound for T'. □

Corollary 4.2.9 (Ivanov, [10])**.** *There is a one-sorted ω-categorical theory which is not G-compact.*

Proof. By theorem 4.1.1, proposition 4.2.8 and the existence of a many-sorted ω-categorical theory with countably many sorts shown in [4]. □

4.3 Classifying T_E

Recall that a theory is unstable iff it has (a formula with) the *strict order property* or (a formula with) the *independence property.*

Recall also that a formula $\varphi(\bar{x}; \bar{y})$ has the *tree property* (with respect to $k < \omega$) if there is a tree of parameters $(\bar{b}_\nu)_{\nu \in \omega^{<\omega}}$ such that every branch is consistent, i.e., for every $\eta \in \omega^\omega$, the set $\{\varphi(\bar{x}; \bar{b}_{\eta \restriction n}) : n < \omega\}$ is consistent and at each node, the branching is k-inconsistent, i.e., for every $\nu \in \omega^{<\omega}$, the set $\{\varphi(\bar{x}; \bar{b}_{\nu^\frown i}) : i < \omega\}$ is k-inconsistent. A theory is not simple iff it has (a formula with) the tree property with respect to some $k < \omega$.

From [20], we say that a formula $\varphi(\bar{x}; \bar{y})$ has the *tree property of the first kind* (TP_1) if there is a tree of parameters $(\bar{b}_\eta)_{\eta \in \omega^{<\omega}}$ such that for each $\eta \in \omega^\omega$, the set $\left\{\varphi(\bar{x} : \bar{b}_{\eta\restriction n}) : n < \omega\right\}$ is consistent and for any two incomparable finite sequences $\nu, \nu' \in \omega^{<\omega}$, the set $\left\{\varphi(\bar{x} : \bar{b}_{\nu}), \varphi(\bar{x} : \bar{b}_{\nu'})\right\}$ is inconsistent. And we say that the formula $\varphi(\bar{x}; \bar{y})$ has the *tree property of the second kind* (TP_2) if there is an array of tuples $(\bar{b}^i_j)_{i,j<\omega}$ such that every row is 2-inconsistent, i.e., for each $i < \omega$ the set $\left\{\varphi(\bar{x}; \bar{b}^i_j) : j < \omega\right\}$ is 2-inconsistent, and any vertical path is consistent, i.e., for each $\eta \in \omega^\omega$, the set $\left\{\varphi(\bar{x}; \bar{b}^i_{\eta(i)}) : i < \omega\right\}$ is consistent.

Theorem III.7.11 of Shelah's book ([19]) states the following.

Theorem 4.3.1. *A theory has the tree property if and only if it has TP_1 or TP_2.*

The *strong order properties* SOP_1 and SOP_2 were defined in [8] in order to find more division lines inside the class of non-simple theories without the strict order property, for which Shelah had already introduced the *strong order properties* SOP_n for $n \geq 3$ in [22]. We say that a theory T has SOP_1 if there is a formula $\varphi(\bar{x}, \bar{y})$ which exemplifies it, i.e., there is a binary tree of parameters $(\bar{a}_\eta)_{\eta\in 2^{<\omega}}$ such that each branch is consistent, i.e, for each $\rho \in 2^\omega$, the set $\{\varphi(\bar{x}, \bar{a}_{\rho\restriction n}) : n \in \omega\}$ is consistent, and whenever $\nu^\frown\langle 0\rangle \subseteq \eta \in 2^{<\omega}$, then $\left\{\varphi(\bar{x}, \bar{a}_\eta), \varphi(\bar{x}, \bar{a}_{\nu^\frown\langle 1\rangle})\right\}$ is inconsistent. Similarly we say that a theory T has SOP_2 if there is a formula $\varphi(\bar{x}, \bar{y})$ which exemplifies it, i.e., there is a binary tree of parameters $(\bar{a}_\eta)_{\eta\in 2^{<\omega}}$ such that any branch is consistent, but whenever $\nu, \eta \in 2^{<\omega}$ are incomparable, then the set $\{\varphi(\bar{x}, \bar{a}_\eta), \varphi(\bar{x}, \bar{a}_\nu)\}$ is inconsistent. It is easy to see that a formula has TP_1 if and only if it has SOP_2, thus TP_2 implies the independence property. From [22] and [8] we know the following fact.

Fact 4.3.2. *For a theory T,*

$$\textit{strict order property} \implies SOP_{n+1} \implies SOP_n \ (\textit{for } n \geq 1)$$

The following definitions are from [8]. Let α be an ordinal. Given two tuples $\bar{\eta}_l = \langle \eta^l_0, \eta^l_1, \ldots, \eta^l_{n_l}\rangle$ $(l = 0, 1)$ of elements of $2^{<\alpha}$, we say that $\bar{\eta}_1 \approx_1 \bar{\eta}_2$ iff $n_0 = n_1$, and the truth values of

i) $\eta^l_{k_3} \subseteq \eta^l_{k_1} \cap \eta^l_{k_2}$

ii) $\eta^l_{k_1} \cap \eta^l_{k_2} \subset \eta^l_{k_3}$

iii) $(\eta^l_{k_1} \cap \eta^l_{k_2})^\frown\langle 0\rangle \subseteq \eta^l_{k_3}$

do not depend on l. We say that a sequence $\langle \bar{a}_\eta : \eta \in 2^{<\alpha}\rangle$ is a *one-full-binary tree indiscernible (1-fbti)* iff whenever $\bar{\eta}_0 \approx_1 \bar{\eta}_1$, then

$$\bar{a}_{\bar{\eta}_0} := \bar{a}_{\eta^0_0} \ldots \bar{a}_{\eta^0_{n_0}} \equiv \bar{a}_{\eta^1_0} \ldots \bar{a}_{\eta^1_{n_0}} := \bar{a}_{\bar{\eta}_1}$$

We will make use of the following fact proved in [8].

Fact 4.3.3. *For any sequence $\langle \bar{b}_\eta : \eta \in 2^{<\omega} \rangle$ and any ordinal $\delta \geq \omega$, we can find $\langle \bar{a}_\eta : \eta \in 2^{<\delta} \rangle$ such that:*

1. *$\langle \bar{a}_\eta : \eta \in 2^{<\delta} \rangle$ is a 1-$fbti$*
2. *If $\bar{\eta} = \langle \eta_m : m < n \rangle$, where each $\eta_m \in 2^{<\omega}$ is given, and Δ is a finite set of formulas from T, then we can find $\nu_m \in 2^{<\omega}$ $(m < n)$ such that if $\bar{\nu} = \langle \eta_m : m < n \rangle$, we have*
   a) *$\bar{\nu} \approx_1 \bar{\eta}$*
   b) *$\bar{a}_{\bar{\eta}} \equiv_\Delta \bar{b}_{\bar{\nu}}$*

Theorem 4.3.4. *T_E is not simple.*

Proof. Observe that the formula $\varphi(x; y, z, w) = E_2(x, y; z, w)$ has TP_2. For convenience in the argument, let $(b_i)_{i\in\omega}, (c_i)_{i\in\omega}, (d_i)_{i\in\omega}$ be three infinite disjoint sequences of different elements such that for every $i \neq j$, $\neg E_2(c_i, d_i; c_j, d_j)$ [2]. For $i, j \in \omega$, let $\bar{a}^i_j = b_i c_j d_j$. By compactness we can see that for any $\eta \in \omega^\omega$, the set $\left\{\varphi(x; \bar{a}^i_{\eta(i)}) : i < \omega\right\}$ is consistent, and since the $c_i d_i$'s are in different E_2-classes, for each $i < \omega$, the set $\left\{\varphi(x; \bar{a}^i_j) : j < \omega\right\}$ is 2-inconsistent. This shows that that $\varphi(x; y, z, w)$ has TP_2. □

Theorem 4.3.5. *T_E does not have SOP. Moreover, T_E does not have SOP$_1$, and therefore T_E does not have TP$_1$.*

Proof. Suppose there is a formula $\varphi(\bar{x}, \bar{y})$, $lg(x) = n$, $lg(y) = m$, and tuples $\langle \bar{a}_\eta : \eta \in 2^{<\omega} \rangle$ in $\mathfrak{C}^m$ which exemplify SOP_1. By fact 4.3.3, we can assume $\langle \bar{a}_\eta : \eta \in 2^{<\omega} \rangle$ is a 1-$fbti$ and by elimination of quantifiers we may also assume that $\varphi(\bar{x}, \bar{y})$ is quantifier free.

Claim. We can assume $\varphi(\bar{x}, \bar{y})$ gives the full diagram of $\bar{x}^\frown\bar{y}$.

proof of the claim. Take a branch, say $\rho \in 2^\omega$, and a realization $\bar{b}_\rho \in \mathfrak{C}^n$ such that $\models \varphi(\bar{b}_\rho, \bar{a}_\nu)$ for any $\nu \subseteq \rho$, $\nu \in 2^{<\omega}$. Consider, for each $\nu \subseteq \rho$, $\nu \in 2^{<\omega}$, the formula $\delta_\nu(\bar{x}, \bar{y})$ given by the conjunction of the quantifier free type of $\bar{b}_\rho^\frown\bar{a}_\nu$ (which is clearly a finite set of formulas). Since there are just finitely many of such formulas, there is a finite sequence $\sigma \subseteq \rho$ such that $\models \delta_\sigma(\bar{b}_\rho, \bar{a}_\nu)$ for infinitely many $\nu \subseteq \rho$, $\nu \in 2^{<\omega}$. We can therefore rename the nodes in the branch in order to have $\models \delta_\sigma(\bar{b}_\rho, \bar{a}_\nu)$ for all $\nu \subseteq \rho$ in $2^{<\omega}$. By indiscernibility, we can assume the same for all branches.

By definition of SOP$_1$, there are $\bar{e} = \langle e^1, \ldots, e^n \rangle$ and $\bar{d} = \langle d^1, \ldots, d^n \rangle$ in $\mathfrak{C}^n$ such that

i) $\mathfrak{C} \models \varphi(\bar{e}, \bar{a}_{\langle\rangle}) \wedge \varphi(\bar{e}, \bar{a}_{\langle 0 \rangle}) \wedge \varphi(\bar{e}, \bar{a}_{\langle 00 \rangle})$

[2] We can do this because there are infinitely many E_2-classes

ii) $\mathfrak{C} \models \varphi(\bar{d}, \bar{a}_{\langle\rangle}) \wedge \varphi(\bar{d}, \bar{a}_{\langle 1\rangle})$

Suppose we can find a model $M_0 \models T_0$ with $\bar{a}_{\langle 00\rangle}, \bar{a}_{\langle 1\rangle}$ in M_0^m and a tuple $\bar{b} = \langle b^1, \ldots, b^n\rangle \in M_0^n$ such that $M_0 \models \varphi(\bar{b}, \bar{a}_{\langle 00\rangle}) \wedge \varphi(\bar{b}, \bar{a}_{\langle 1\rangle}) \wedge p(\bar{a}_{\langle 00\rangle}, \bar{a}_{\langle 1\rangle})$, where $p(\bar{z}_1; \bar{z}_2) = \mathrm{qftp}_{\mathfrak{C}}(\bar{a}_{\langle 00\rangle}, \bar{a}_{\langle 1\rangle})$. Since $\varphi(\bar{x}, \bar{y})$ is quantifier free, we could find a model $M \models T, M_0 \subseteq M$ such that $M \models \varphi(\bar{b}, \bar{a}_{\langle 00\rangle}) \wedge \varphi(\bar{b}, \bar{a}_{\langle 1\rangle}) \wedge p(\bar{a}_{\langle 00\rangle}, \bar{a}_{\langle 1\rangle})$, and embed it elementarily inside $\mathfrak{C}$. This will give us a contradiction with the definition of SOP_1.
We now construct the model M_0. Let $A = \left\{a^1_{\langle 00\rangle}, \ldots, a^m_{\langle 00\rangle}\right\}$, $B = \left\{a^1_{\langle 1\rangle}, \ldots, a^m_{\langle 1\rangle}\right\}$, $C = \left\{b^1, \ldots, b^n\right\}$, and let $M_0 = A \cup B \cup C$ be its universe. To interpret $E_1, \ldots, E_{2m+n}$ in M_0 we do the following. For each $1 \leq r \leq 2m+n$, let $\Gamma_r \subseteq M_0^r \times M_0^r$ be given by $\Gamma_r = D_r \cup E_r^{AB} \cup E_r^{AC} \cup E_r^{BC}$, where:

i) D_r is the diagonal in $M_0^r \times M_0^r$.

ii) $E_r^{AB} = E_r^{\mathfrak{C}} \restriction AB \subseteq (AB)^r \times (AB)^r$.

iii) $E_r^{AC} \subseteq (AC)^r \times (AC)^r$ is the set of all tuples of the form

$$\langle a^{i_1}_{\langle 00\rangle} \ldots a^{i_k}_{\langle 00\rangle} b^{i_{k+1}} \ldots b^{i_r} \ ; \ a^{j_1}_{\langle 00\rangle} \ldots a^{j_l}_{\langle 00\rangle} b^{i_{l+1}} \ldots b^{j_r}\rangle,$$

whenever

$$\mathfrak{C} \models a^{i_1}_{\langle 00\rangle} \ldots a^{i_k}_{\langle 00\rangle} e^{i_{k+1}} \ldots e^{i_r} E_r a^{j_1}_{\langle 00\rangle} \ldots a^{j_l}_{\langle 00\rangle} e^{i_{l+1}} \ldots e^{j_r}$$

And all their possible permutations.

iv) $E_r^{BC} \subseteq (BC)^r \times (BC)^r$ is the set of all tuples of the form

$$\langle a^{i_1}_{\langle 1\rangle} \ldots a^{i_k}_{\langle 1\rangle} b^{i_{k+1}} \ldots b^{i_r} \ ; \ a^{j_1}_{\langle 1\rangle} \ldots a^{j_l}_{\langle 1\rangle} b^{i_{l+1}} \ldots b^{j_r}\rangle,$$

whenever

$$\mathfrak{C} \models a^{i_1}_{\langle 1\rangle} \ldots a^{i_k}_{\langle 1\rangle} d^{i_{k+1}} \ldots d^{i_r} E_r a^{j_1}_{\langle 1\rangle} \ldots a^{j_l}_{\langle 1\rangle} d^{i_{l+1}} \ldots d^{j_r}$$

An all their possible permutations.

We want to extend Γ_r to an equivalence relation on M_0^r, say $E_r^{M_0}$, such that:

- $E_r^{M_0} \restriction AC = E_r^{AC}$
- $E_r^{M_0} \restriction BC = E_r^{BC}$
- $E_r^{M_0} \restriction AB = E_r^{AB}$

Let $E_r^{M_0}$ be the transitive closure of Γ_r. It is clearly an equivalence relation on M_0^r. To check the additional previous conditions it's enough to prove that:

a) $(E_r^{AC} \circ E_r^{AB} \circ E_r^{BC}) \restriction AC \subseteq E_r^{AC}$

b) $(E_r^{AC} \circ E_r^{AB}) \restriction BC \subseteq E_r^{BC}$

c) $(E_r^{AC} \circ E_r^{BC}) \restriction AB \subseteq E_r^{AB}$

It is easy to see that these are the main possible cases of getting new tuples from $(AB)^r, (AC)^r, (BC)^r$ by composing Γ_r with itself finitely many times. The other cases are either analogous or can be reduced to one of them. We check these cases:

a) Assume

$$\begin{array}{rcl} \langle b^{i_1} \dots b^{i_s} a^{i_{s+1}}_{\langle 00 \rangle} \dots a^{i_r}_{\langle 00 \rangle} \ ; \ a^{j_1}_{\langle 00 \rangle} \dots a^{j_r}_{\langle 00 \rangle} \rangle & \in & E_r^{AC}, \\ \langle a^{j_1}_{\langle 00 \rangle} \dots a^{j_r}_{\langle 00 \rangle} \ ; \ a^{k_1}_{\langle 1 \rangle} \dots a^{k_r}_{\langle 1 \rangle} \rangle & \in & E_r^{AB}, \\ \langle a^{k_1}_{\langle 1 \rangle} \dots a^{k_r}_{\langle 1 \rangle} \ ; \ b^{l_1} \dots b^{l_r} \rangle & \in & E_r^{BC}. \end{array}$$

We want to see that $\langle b^{i_1} \dots b^{i_s} a^{i_{s+1}}_{\langle 00 \rangle} \dots a^{i_r}_{\langle 00 \rangle} \ ; \ b^{l_1} \dots b^{l_r} \rangle \in E_r^{AC}$, and for this it's enough to check that $\mathfrak{C} \models e^{i_1} \dots e^{i_s} a^{i_{s+1}}_{\langle 00 \rangle} \dots a^{i_r}_{\langle 00 \rangle} E_r e^{l_1} \dots e^{l_r}$. We know that

$$\varphi(\bar{x}, \bar{y}) \models x^{i_1} \dots x^{i_s} y^{i_{s+1}} \dots y^{i_r} E_r y^{j_1} \dots y^{j_r} \tag{4.1}$$

$$\varphi(\bar{x}, \bar{y}) \models y^{k_1} \dots y^{k_r} E_r x^{l_1} \dots x^{l_r} \tag{4.2}$$

Since $\mathfrak{C} \models \varphi(\bar{e}, \bar{a}_{\langle \rangle}) \wedge \varphi(\bar{e}, \bar{a}_{\langle 0 \rangle}) \wedge \varphi(\bar{e}, \bar{a}_{\langle 00 \rangle})$ and $\mathfrak{C} \models \varphi(\bar{d}, \bar{a}_{\langle \rangle}) \wedge \varphi(\bar{d}, \bar{a}_{\langle 1 \rangle})$, we know that in $\mathfrak{C}$,

$$\begin{array}{rcl} e^{i_1} \dots e^{i_s} a^{i_{s+1}}_{\langle 00 \rangle} \dots a^{i_r}_{\langle 00 \rangle} \mathbf{E_r} a^{j_1}_{\langle 00 \rangle} \dots a^{j_r}_{\langle 00 \rangle} \mathbf{E_r} a^{k_1}_{\langle 1 \rangle} \dots a^{k_r}_{\langle 1 \rangle} & \mathbf{E_r} & d^{l_1} \dots d^{l_r} \\ & \mathbf{E_r} & a^{k_1}_{\langle \rangle} \dots a^{k_r}_{\langle \rangle} \\ & \mathbf{E_r} & e^{l_1} \dots e^{l_r}, \end{array}$$

as we wanted.

b) Assume $\langle b^{i_1} \dots b^{i_r} ; a^{j_1}_{\langle 00 \rangle} \dots a^{j_r}_{\langle 00 \rangle} \rangle \in E_r^{AC}$ and $\langle a^{j_1}_{\langle 00 \rangle} \dots a^{j_r}_{\langle 00 \rangle} ; a^{k_1}_{\langle 1 \rangle} \dots a^{k_r}_{\langle 1 \rangle} \rangle \in E_r^{AB}$. We want to see that $\langle b^{i_1} \dots b^{i_r} \ ; \ a^{k_1}_{\langle 1 \rangle} \dots a^{k_r}_{\langle 1 \rangle} \rangle \in E_r^{BC}$, and for this it's enough to check that $\mathfrak{C} \models d^{i_1} \dots d^{i_r} E_r a^{k_1}_{\langle 1 \rangle} \dots a^{k_r}_{\langle 1 \rangle}$. Since $\varphi(\bar{x}, \bar{y}) \models x^{i_1} \dots x^{i_r} E_r y^{j_1} \dots y^{j_r}$, we know that in $\mathfrak{C}$,

$$d^{i_1} \dots d^{i_r} \mathbf{E_r} a^{j_1}_{\langle \rangle} \dots a^{j_r}_{\langle \rangle} \mathbf{E_r} e^{i_1} \dots e^{i_r} \mathbf{E_r} a^{j_1}_{\langle 00 \rangle} \dots a^{j_r}_{\langle 00 \rangle} \mathbf{E_r} a^{k_1}_{\langle 1 \rangle} \dots a^{k_r}_{\langle 1 \rangle},$$

as we wanted.

c) Assume $\langle b^{i_1} \dots b^{i_r} \ ; \ a^{j_1}_{\langle 00 \rangle} \dots a^{j_r}_{\langle 00 \rangle} \rangle \in E_r^{AC}$ and $\langle b^{i_1} \dots b^{i_r} \ ; \ a^{k_1}_{\langle 1 \rangle} \dots a^{k_r}_{\langle 1 \rangle} \rangle \in E_r^{BC}$. We want to see that $\langle a^{j_1}_{\langle 00 \rangle} \dots a^{j_r}_{\langle 00 \rangle} \ ; \ a^{k_1}_{\langle 1 \rangle} \dots a^{k_r}_{\langle 1 \rangle} \rangle \in E_r^{AB}$, and for this it's enough to check that $\mathfrak{C} \models a^{j_1}_{\langle 00 \rangle} \dots a^{j_r}_{\langle 00 \rangle} E_r a^{k_1}_{\langle 1 \rangle} \dots a^{k_r}_{\langle 1 \rangle}$. We know that

$$\varphi(\bar{x}, \bar{y}) \models x^{i_1} \dots x^{i_r} E_r y^{j_1} \dots y^{j_r} \tag{4.3}$$

$$\varphi(\bar{x}, \bar{y}) \models x^{i_1} \dots x^{i_r} E_r y^{k_1} \dots y^{k_r} \tag{4.4}$$

Since $\mathfrak{C} \models \varphi(\bar{e}, \bar{a}_{\langle\rangle}) \wedge \varphi(\bar{e}, \bar{a}_{\langle 0\rangle}) \wedge \varphi(\bar{e}, \bar{a}_{\langle 00\rangle})$ and $\mathfrak{C} \models \varphi(\bar{d}, \bar{a}_{\langle\rangle}) \wedge \varphi(\bar{d}, \bar{a}_{\langle 1\rangle})$, we know that in $\mathfrak{C}$,

$$a^{j_1}_{\langle 00\rangle} \dots a^{j_r}_{\langle 00\rangle} \mathbf{E_r} e^{i_1} \dots e^{i_r} \mathbf{E_r} a^{j_1}_{\langle\rangle} \dots a^{j_r}_{\langle\rangle} \mathbf{E_r} a^{k_1}_{\langle\rangle} \dots a^{k_r}_{\langle\rangle} \mathbf{E_r} d^{i_1} \dots d^{i_r} \mathbf{E_r} a^{k_1}_{\langle 1\rangle} \dots a^{k_r}_{\langle 1\rangle},$$

as we wanted.

Once the E_r's are interpreted, we get the model M_0 we wanted and therefore the contradiction. Since SOP_2 implies SOP_1, T_E does not have TP_1. $\square$

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