NIP formulas and theories

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September 19, 2010. Revised February 16, 2011 *

T is a complete theory with infinite models, L is its language and $\mathfrak{C}$ is its monster model.

1 Formulas with IP

Definition 1.1 $\varphi(x, y)$ has IP (*the independence property*) if there are $(a_i : i < \omega)$, $(b_I : I \subseteq \omega)$ such that

$$\models \varphi(a_i, b_I) \Leftrightarrow i \in I$$

If φ does not have IP we say it has NIP. It is said that T has IP if some formula has IP in T, and otherwise it is said that T has NIP.

Remark 1.2 1. *If φ has* IP*, then for every set X there are $(a_i : i \in X)$, $(b_I : I \subseteq X)$ such that $\models \varphi(a_i, b_I) \Leftrightarrow i \in I$.*

2. *If for arbitrarily large $n < \omega$ there are $(a_i : i < n)$ such that for all $I \subseteq n$,*

$$\{\varphi(a_i, y) : i \in I\} \cup \{\neg\varphi(a_i, y) : i \in n \smallsetminus I\}$$

is consistent, then $\varphi(x, y)$ has IP.

3. *If $\varphi(x, y, z) \in L$ and $\varphi(x, y, a)$ has* IP *in $T(a)$, then $\varphi(x; y, z)$ has* IP *in T.*

Lemma 1.3 *If $\varphi(x, y)$ has* IP*, then $\varphi^{-1}(y, x)$ has* IP.

Proof: Let $n < \omega$. There are $(a_X : X \in \mathcal{P}(n))$, $(b_I : I \subseteq \mathcal{P}(n))$ such that $\models \varphi(a_X, b_I) \Leftrightarrow X \in I$. Let $U_i := \{X \subseteq n : i \in X\}$ for $i < n$ and let $c_i := b_{U_i}$. Then $\models \varphi^{-1}(c_i, a_X) \Leftrightarrow i \in X$. □

Definition 1.4 The *alternation number* of $\varphi(x, y)$, $\mathrm{alt}(\varphi)$, is the maximal n such that for some indiscernible sequence $(a_i : i < \omega)$, for some b, ω can be decomposed in consecutive segments $I_1, \ldots, I_n$, and $\varphi(a_i, b)$ has constant truth value for i in the same segment and opposite truth value to $\varphi(a_j, b)$ if i, j are in consecutive segments. If the maximal n does not exists we put $\mathrm{alt}(\varphi) = \infty$.

Remark 1.5 1. *If $\mathrm{alt}(\varphi) = \infty$, then for every ordinal α there is an indiscernible sequence $(a_i : i < \alpha)$ such that for some b, for all $i < \alpha$, $\models \varphi(a_i, b) \leftrightarrow \neg\varphi(a_{i+1}, b)$.*

*Notes of talks given at the Model Theory Seminar in the academic year 2009-2010. Thanks to all members of the Seminar for their comments: Santiago Cárdenas, Rafel Farré, Magnus Kollmann, Martin Koerwien, Daniel Palacín, Juan Francisco Pons, and Joris Potier. Thanks also to the visiting members of the Seminar: John Baldwin, Sy Friedman, and Margarita Otero.

2. *If* $\mathrm{alt}(\varphi) < \infty$, *then for all limit ordinals* α, *for all indiscernible sequences* $(a_i : i < \alpha)$, *for all* b, *there is some* $j < \alpha$ *such* $\models \varphi(a_i, b)$ *for all* $i > j$ *or* $\models \neg\varphi(a_i, b)$ *for all* $i > j$.

Proposition 1.6 φ *has* IP *if and only if* $\mathrm{alt}(\varphi) = \infty$

Proof: Assume $\varphi(x, y)$ has IP. There is some indiscernible sequence $(a_i : i < \omega)$ such that for all $I \subseteq \omega$ the set $\{\varphi(a_i, y) : i \in I\} \cup \{\neg\varphi(a_i, y) : i \in \omega \smallsetminus I\}$ is consistent. In particular, $\{\varphi(a_{2\cdot i}, y) : i < \omega\} \cup \{\neg\varphi(a_{2\cdot i+1}, y) : i < \omega\}$ is consistent, which clearly implies $\mathrm{alt}(\varphi) = \infty$.

For the other direction, assume $\mathrm{alt}(\varphi) = \infty$ and choose an indiscernible sequence $(a_i : i < \omega)$ such that $\{\varphi(a_{2\cdot i}, y) : i < \omega\} \cup \{\neg\varphi(a_{2\cdot i+1}, y) : i < \omega\}$ is consistent. We claim that $\{\varphi(a_i, y) : i \in I\} \cup \{\neg\varphi(a_i, y) : i \in \omega \smallsetminus I\}$ is consistent for all $I \subseteq \omega$. It is enough to check that for any finite disjoint $I, J \subseteq \omega$, $\Sigma(y) := \{\varphi(a_i, y) : i \in I\} \cup \{\neg\varphi(a_i, y) : i \in J\}$ is consistent. Let $m_1 < \ldots < m_n$ and $k_1 < \ldots < k_j$ be respective enumerations of X and Y and choose even numbers $m'_1 < \ldots < m'_n$ and odd numbers $k'_1 < \ldots < k'_j$ such that $m_1, \ldots, m_n, k_1, \ldots, k_j$ and $m'_1, \ldots, m'_n, k'_1, \ldots, k'_j$ have the same order type. By assumption $\{\varphi(a_i, y) : i = m'_1, \ldots, m'_n\} \cup \{\neg\varphi(a_i, y) : i = k'_1, \ldots, k'_j\}$ is consistent. By indiscernibility $\Sigma(y)$ is consistent. □

Remark 1.7 *Every boolean combination* $\varphi(x_1, \ldots, x_n; y_1, \ldots, y_n)$ *of formulas* $\varphi_i(x_i, y_i)$ *with* NIP *has also* NIP. *The tuple* x_i *may have elements in common with* x_j *but it is disjoint with* y_j.

Proof: We may assume $x_i = x_j$ and $y_i = y_j$ for all i, j. It is clear that $\neg\varphi(x, y)$ has IP if and only if $\varphi(x, y)$ has IP. On the other hand, an easy argument shows that if $\varphi(x; y) := \varphi_1(x, y) \wedge \varphi_2(x, y)$ has infinite alternation number then one of the formulas $\varphi_i(x, y)$ has also infinite alternation number. □

Proposition 1.8 *Let* y *be a fixed* n*-tuple of variables. The following are equivalent:*

1. *No formula* $\varphi(x, y)$ *has* IP.
2. *If* α *has cofinality* $\geq |T|^+$, $(a_i : i < \alpha)$ *is an indiscernible sequence, and* B *is a set of* $\leq n$ *elements, then for some* $j < \alpha$, $(a_i : j < i < \alpha)$ *is* B*-indiscernible.*

Proof: *1* $\Rightarrow$ *2* It is enough to prove, assuming *1*, that for each $\varphi(x_1, \ldots, x_m; y) \in L$, for each limit ordinal α, for each indiscernible sequence $(a_i : i < \alpha)$, for each n-tuple b, there is some $j < \alpha$ such that for all tuples $j < i_1 < \ldots < i_m < \alpha$, for all $j < l_1 < \ldots < l_m < \alpha$, $\models \varphi(a_{i_1}, \ldots, a_{i_m}; b) \leftrightarrow \varphi(a_{l_1}, \ldots, a_{l_m}; b)$. And this is the case, since otherwise the indiscernible sequence $(b_i : i < \omega)$ with $b_i := a_{0\cdot i}, \ldots, a_{(m-1)\cdot i}$ would witness that $\mathrm{alt}(\varphi(x_1, \ldots, x_m; y)) = \infty$.

2 $\Rightarrow$ *1* is clear by point *1* in Remark 1.5. □

Proposition 1.9 *If some formula has* IP *in* T, *there is some* $\varphi(x, y)$ *with* IP *where* x *is a single variable.*

Proof: By Lemma 1.3 it suffices to find some IP formula $\varphi(x, y)$ where y is a single variable. This follows from Proposition 1.8 since point *2* for all $|B| \leq n$ implies point *2* for all $|B| \leq n + 1$. We check this. Assume $B = \{b_1, \ldots, b_{n+1}\}$, α has cofinality $\geq |T|^+$ and $(a_i : i < \alpha)$ is an indiscernible sequence such that for no $j < \alpha$ the sequence $(a_i : j < i < \alpha)$ is B-indiscernible. Choose $j < \alpha$ such that $(a_i : j < i < \alpha)$ is b_{n+1}-indiscernible. Then $(a_i b_{n+1} : j < i < \alpha)$ is indiscernible and we can choose now some $l < \alpha$ such that $j \leq l$ and $(a_i b_{n+1} : l < i < \alpha)$ is $\{b_1, \ldots, b_n\}$-indiscernible. It follows that $(a_i : l < i < \alpha)$ is B-indiscernible. □

2 Number of types

Definition 2.1 For any cardinal κ, $\mathrm{ded}(\kappa)$ is the supremum of the number of branches of a tree of cardinality κ.

Remark 2.2 *1. $\mathrm{ded}(\kappa)$ is the supremum of the cardinalities of linearly ordered sets having a dense subset of cardinality κ.*

2. $\kappa^\omega \leq \mathrm{ded}(\kappa) \leq 2^\kappa$.

Proof: *1.* Given a tree, consider the lexicographic order of nodes and branches. Given a linearly ordered set construct a convenient set of closed intervals with endpoints in the dense set.

2. For $\kappa^\omega \leq \mathrm{ded}(\kappa)$ note that κ^ω can be identified with the set of branches of the tree $\kappa^{<\omega}$. □

Lemma 2.3 *If $F \subseteq 2^\lambda$ and $|F| > \mathrm{ded}(\lambda)$, then for each $n < \omega$ there is some $I \subseteq \lambda$ such that $|I| = n$ and $F \upharpoonright I = 2^I$.*

Proof: Assume F, λ are a counterexample, with λ minimal. Note that F can be naturally identified with a set of branches of the tree $\bigcup_{i<\lambda} F \upharpoonright i$. By minimality of λ, we may assume that for each $i < \lambda$, $|F \upharpoonright i| \leq \mathrm{ded}(\lambda)$.

For each $f \in F \upharpoonright i$, let $F(f) = \{g \in F : f \subseteq g\}$, let $G_i = \{f \in F \upharpoonright i : |F(f)| > \mathrm{ded}(\lambda)$, and let $G = \{f \in F : f \upharpoonright i \in G_i \text{ for all } i < \lambda\}$. Then $G \subseteq F$ is a set of branches of the tree $\bigcup_{i<\lambda} G_i$. Note that $F \smallsetminus G = \bigcup_{i<\lambda} \bigcup_{g \in F \upharpoonright i \smallsetminus G_i} F(g)$ and hence $|F \smallsetminus G| \leq \mathrm{ded}(\lambda)$, and $|G| > \mathrm{ded}(\lambda)$. Therefore, we can assume $G = F$. In other terms, we can assume that for each $i < \lambda$, $|F(f)| > \mathrm{ded}(\lambda)$.

Now we prove by induction on n, that for each $n < \omega$, for each $f \in \bigcup_{i<\lambda} F \upharpoonright i$ there is some $I \subseteq \lambda$ such that $|I| = n$ and $F(f) \upharpoonright I = 2^I$. This is clear for $n = 0$ since $F(f) \neq \emptyset$. Let us consider the case $n+1$. By definition of ded, since $F(f)$ is a set of branches of the tree $\bigcup_{i<\lambda} F(f) \upharpoonright i$, this tree has cardinality $> \lambda$ and therefore $|F(f) \upharpoonright i| > \lambda$ for some $i < \lambda$. By the induction hypothesis, for each $g \in F(f) \upharpoonright i$ there is some $I_g \subseteq \lambda$ such that $|I_g| = n$ and $F(g) \upharpoonright I_g = 2^{I_g}$. By cardinality reasons, there are two different $g, h \in F(f) \upharpoonright i$ such that $I := I_g = I_h$. Choose $j < i$ such that $h(j) \neq g(j)$. Then $j \notin I$. If $J = I \cup \{j\}$, then $F(f) \upharpoonright J = 2^J$. □

Proposition 2.4 *1. If φ has* IP*, then for each cardinal κ there is a set A of cardinality κ such that $|S_\varphi(A)| = 2^\kappa$.*

2. If φ has NIP*, then for each cardinal κ: if $|A| = \kappa$, then $|S_\varphi(A)| \leq \mathrm{ded}(\kappa)$.*

Proof: *1* is clear. For *2* we use Lemma 2.3. Assume $|A| = \kappa$ and $|S_\varphi(A)| > \mathrm{ded}(\kappa)$. Let $\varphi = \varphi(x, y)$ and let l be the length of y. Fix an enumeration $(a_i : i < \kappa)$ of A^l. For each $p(x) \in S_\varphi(A)$, let $f_p \in 2^\kappa$ be defined by $f_p(i) = 0$ iff $\varphi(x, a_i) \in p$. Let $F = \{f_p : p \in S_\varphi(A)\}$. Since $|F| > \mathrm{ded}(\kappa)$, for each $n < \omega$ there is some $I \subseteq \kappa$ such that $|I| = n$ and $F \upharpoonright I = 2^I$. For each $X \subseteq I$, $\{\varphi(x, a_i) : i \in X\} \cup \{\neg\varphi(x, a_i) : i \in I \smallsetminus X\}$ is consistent. Hence $\varphi(x, y)$ has IP. □

Corollary 2.5 *1. If T has* IP*, then for each cardinal $\kappa \geq |T|$ there is a set A of cardinality κ such that $|S_1(A)| = 2^\kappa$.*

2. *If φ has* NIP, *then for each cardinal $\kappa \geq \omega$: if $|A| = \kappa$, then $|S_n(A)| \leq \text{ded}(\kappa)^{|T|}$.*

Remark 2.6 *$\kappa < \text{ded}(\kappa)$ for all infinite κ.*

Proof: Assume $\kappa = \text{ded}(\kappa)$. This implies every NIP formula is stable, which is not true. If $\varphi(x, y)$ is NIP, then for each set A of cardinality $\leq \kappa$, $|S_\varphi(A)| \leq \text{ded}(\kappa) = \kappa$ and hence φ is κ-stable and therefore stable. □

3 Stability and simplicity

The reader is assumed to be familiar with the following definitions and the facts concerning stability and simplicity stated thereafter. See [7] for details.

Definition 3.1 (Reminding)

1. $\varphi(x, y)$ is *stable* if for all infinite λ, for all A, if $|A| \leq \lambda$, then $|S_\varphi(A)| \leq \lambda$. Otherwise it is *unstable*.
2. $\varphi(x, y)$ has the *order property* if there are $(a_i : i < \omega)$ and $(b_i : i < \omega)$ such that
$$\models \varphi(a_i, b_j) \Leftrightarrow i < j$$
3. $\varphi(x, y)$ has the *strict order property* if there are $(a_i : i < \omega)$ such that $\varphi(\mathfrak{C}, a_i) \subsetneq \varphi(\mathfrak{C}, a_{i+1})$.
4. T is *stable* if all formulas are stable in T. Otherwise it is *unstable*.

Fact 3.2 *(Reminding)*

1. *Stable formulas are* NIP.
2. *If T is unstable, there is an unstable formula $\varphi(x, y)$ where x is a single variable.*
3. *$\varphi(x, y)$ is stable if and only if it does not have the order property.*
4. *If φ is stable, then also φ^{-1} is stable.*
5. *Let y be a n-tuple of variables. If $\varphi(x, y)$ has the strict order property, then*
$$\psi(y_1, y_2) := \forall x(\varphi(x, y_1) \to \varphi(x, y_2)) \wedge \exists x(\varphi(x, y_2) \wedge \neg\varphi(x, y_1))$$
defines a partial order of $\mathfrak{C}^n$ which has infinite chains.
6. *If $\varphi(x, y)$ is unstable and it is* NIP, *then*
   (a) *Some conjunction of $\varphi(x, y)$ with formulas of the form $\varphi(x, a)$ and $\neg\varphi(x, a)$ has the strict order property.*
   (b) *For some $n < \omega$, for some $s \in 2^n$, the formula $\psi(x; y_1, \ldots, y_n) = \bigwedge_{i<n} \varphi(x, y_{i+1})^{s(i)}$ has the strict order property (where $\varphi^0 = \varphi$ and $\varphi^1 = \neg\varphi$).*
7. *T is unstable if and only if it has* IP *or it has the strict order property.*

Proposition 3.3 *Assume T is an unstable* NIP *theory. Then there is a definable partial order of the universe $\mathfrak{C}$ with infinite chains.*

Proof: Choose an unstable $\varphi(x, y)$ where y is a single variable and apply points *6 (a)* and *5* of Proposition 3.2. $\square$

Definition 3.4 (Reminding)

1. $\varphi(x, y)$ has the *k–tree property* if there is a tree $(a_s : s \in \omega^{<\omega})$ such that $\{\varphi(x, a_{f\restriction n}) : n < \omega\}$ is consistent for every $f \in \omega^\omega$, and $\{\varphi(x, a_{s^\frown i}) : i < \omega\}$ is k-inconsistent for every $s \in \omega^{<\omega}$.

2. T is *simple* if no formula has the k–tree property in T for any $k < \omega$.

Fact 3.5 *(Reminding)*

1. *If T is not simple, some formula $\varphi(x, y)$, where x is a single variable, has the 2-tree property in T.*

2. *Stable formulas do not have the k-tree property.*

3. *Stable theories are simple.*

4. *If $\varphi(x, y)$ has the strict order property, then*

$$\psi(x; y_1 y_2) := \neg\varphi(x, y_1) \wedge \varphi(x, y_2)$$

has the 2-tree property.

5. *Simple theories do not have the strict order property.*

6. *Simple unstable theories have the* IP.

7. *Stable theories are just those simple theories that have* NIP.

4 O-minimality

Definition 4.1 T is *o-minimal* if the language of T contains a binary predicate $<$ which is interpreted as a linear order of the universe and every definable set is a finite union of open intervals ((a, b), $(-\infty, b)$, $(a, +\infty)$, $(-\infty, +\infty)$) and points.

Proposition 4.2 *Every o-minimal theory is NIP.*

Proof: By Proposition 1.9 it is enough to prove that all formulas of the form $\varphi(x, y)$, where x is a single variable, are NIP. Assume $\varphi(x, y)$ is a counterexample to this. By Proposition 1.6, there is some indiscernible sequence of elements $(a_i : i < \omega)$ and some tuple b such that $\models \varphi(a_i, b) \leftrightarrow \neg\varphi(a_{i+1}, b)$. By o-minimality $\varphi(x, b)$ defines a finite union of interval and points. Hence, for some boolean combination $\psi(x, z)$ of formulas $x < z_i$, for some tuple c, $\varphi(x, b)$ is equivalent to $\psi(x, c)$. Since $\models \psi(a_i, c) \leftrightarrow \neg\psi(a_{i+1}, c)$, $\psi(x, z)$ is IP too. By Remark 1.7, $x < z_i$ is IP. But this contradicts point *1* of Proposition 2.4 since for each finite A, $|S_{x<y}(A)| \leq 2 \cdot |A| + 1$. $\square$

Proposition 4.3 *Assume T is o-minimal. If f is a definable function defined on an open interval (a,b), there is a finite sequence $a = a_0 < a_1 \ldots < a_n = b$ such that in every interval (a_i, a_{i+1}) f is constant or strictly increasing or strictly decreasing.*

Proof: See, for instance, Théorème 4.6 in [12] or Theorem 4.2 in [14]. □

Proposition 4.4 *If T is o-minimal, then the operator* $\mathrm{acl} = \mathrm{dcl}$ *has the exchange property and therefore it defines a pregeometry.*

Proof: The order $<$ allows us to define over A all elements of $\mathrm{acl}(A)$ and hence $\mathrm{dcl}(A) = \mathrm{acl}(A)$. We check the exchange property. Assume $b \in \mathrm{acl}(Ac) \smallsetminus \mathrm{acl}(A)$. Then $b \in \mathrm{dcl}(Ac)$ and for some n there is a 0-definable mapping $f : \mathfrak{C}^n \to \mathfrak{C}$ and some $a_1, \ldots, a_n$ such that $f(a_1, \ldots, a_n) = b$. Since $b \notin \mathrm{acl}(A)$, $c = a_i$ for some i. Without loss of generality, $i = 1$. We may assume there is an open interval (c_1, c_2) containing c. By Proposition 4.3, there are $c_1 = d_0 < \ldots < d_m = c_2$ such that f is constant or strictly increasing or strictly decreasing in every interval (d_i, d_{i+1}). We may assume f is not constant nor strictly increasing or decreasing in (d_i, d_{i+2}) and therefore each d_i is definable over A. It follows that $c \neq d_i$ for all i and hence $c \in (d_i, d_{i+1})$ for some i. If f is constant in (d_i, d_{i+1}) then b is A-definable. Assume f is strictly increasing or decreasing in (d_i, d_{i+1}). Then f is one-to-one and hence c is definable over Ab and therefore $c \in \mathrm{acl}(Ab)$. □

5 TP_1 and TP_2

Definition 5.1 $\varphi(x,y)$ has *the tree property of the first kind* (TP_1) if there is a tree $(a_s : s \in \omega^{<\omega})$ such that $\{\varphi(x, a_{f\restriction n}) : n < \omega\}$ is consistent for all $f \in \omega^\omega$, and $\varphi(x, a_s) \wedge \varphi(x, a_t)$ is inconsistent if $s, t \in \omega^{<\omega}$ are incomparable in the lexicographic order. We say that $\varphi(x,y)$ has NTP_1 if it does not have TP_1. The theory T has TP_1 if some formula has TP_1 in T. Otherwise T has NTP_1.

Let $2 \leq k < \omega$. We say that $\varphi(x,y)$ has *the k–tree property of the second kind* if there are a^i_j $(i,j < \omega)$ such that $\{\varphi(x, a^i_{f(i)}) : i < \omega\}$ is consistent for all $f \in \omega^\omega$, and $\{\varphi(x, a^i_j) : j < \omega\}$ is k–inconsistent for all $i < \omega$. The formula $\varphi(x,y)$ has TP_2 if it has the 2-tree property of the second kind. Otherwise it has NTP_2. The theory T has TP_2 if some formula has TP_2 in T. Otherwise it has NTP_2.

Proposition 5.2 *1. If $\varphi(x,y)$ has TP_1, then it has the 2-tree property.*

2. If $\varphi(x,y)$ has the k–tree property of the second kind, then it has the k-tree property.

3. Simple theories have NTP_1 and NTP_2.

4. If $\varphi(x,y)$ has TP_2, then $\varphi(x,y)$ has IP.

5. NIP *theories have NTP_2.*

Proof: *1* is clear. For *2* put $b_\emptyset = a^0_0$ and $b_s = a^{n+1}_{s(n)}$ if $s \in \omega^{n+1}$. *3* follows from *1* and *2*.

4. Let $(a^i_j : i,j < \omega)$ witness TP_2 of $\varphi(x,y)$. We check that $(a^i_0 : i < \omega)$ witnesses IP of $\varphi(x,y)$. Let $X \subseteq \omega$, and let $f : \omega \to \omega$ be defined by $f(n) = 0$ if $n \in X$ and $f(n) = 1$ otherwise. Since $\{\varphi(x, a^i_{f(i)}) : i < \omega\}$ is consistent, $\{\varphi(x, a^n_0) : n \in X\} \cup \{\neg\varphi(x, a^n_0) : n \in \omega \smallsetminus X\}$ is also consistent. *5* follows from *4*. □

Proposition 5.3 *T is simple if and only if it has NTP_1 and NTP_2.*

Proof: Later. □

6 Limit and average types

Definition 6.1 For any sequence $a = (a_i : i \in I)$ of tuples of the same length, if F is a proper filter over I, then we define

$$\lim_F(a/A) := \{\varphi(x) \in L(A) : \{i \in I : \models \varphi(a_i) \in F\}\}.$$

It is a partial type over A and it is finitely satisfiable in $\{a_i : i \in I\}$. If F is an ultrafilter, it is complete: $\lim_F(a/A) \in S(A)$.

If I is a linearly ordered set without last element, $\mathrm{Av}(a/A)$ is $\lim_F(a/A)$ where F is the proper filter over I generated by all (nonempty) final segments. It is called the *average type of a over A*.

There is another use of the notion of average type. If I is an infinite indiscernible set (not just a sequence!), then $\mathrm{Av}(I/A)$ is defined as the partial type over A consisting in all formulas $\varphi(x) \in L(A)$ which are true of almost all $a \in I$, i.e., $|\{a \in I : \models \neg\varphi(a)\}| < \omega$. Note that in this sense $\mathrm{Av}(I/A) = \lim_F(a/A)$ if a is a one-to-one enumeration of I and F is the filter of all cofinite subsets of the index set. We will see that in NIP theories these two notions of average type are compatible.

All this applies also when $A = \mathfrak{C}$.

Proposition 6.2 *Assume T has* NIP.

1. *Let I be a linearly ordered set without last element, and let $a = (a_i : i \in I)$ be indiscernible. Then $\mathrm{Av}(a/A) \in S(A)$.*
2. *Assume I is an infinite indiscernible set. Then $\mathrm{Av}(I/A) \in S(A)$. If a is an enumeration of I with an index set linearly ordered without last element, then $\mathrm{Av}(I/A) = \mathrm{Av}(a/A)$.*

Proof: *1* follows from Proposition 1.6. *2*. Assume first $|I| = \omega$ and let a be an enumeration of I of order type ω. In this case it is clear, by definition, that $\mathrm{Av}(I/A) = \mathrm{Av}(a/A)$. From this it follows (by considering a suitable countable subset) that for any infinite indiscernible set I, $\mathrm{Av}(I/A) \in S(A)$. Now let a be an enumeration of I by a linear ordering without last element. Clearly $\mathrm{Av}(I/A) \subseteq \mathrm{Av}(a/A)$. Since they are complete types, they coincide. □

Remark 6.3 *If $\varphi(x, y)$ has* NIP*, then there is some $k < \omega$ such that for every b, for every infinite indiscernible set I, either $|\{a \in I : \models \varphi(a, b)\}| < k$ or $|\{a \in I : \models \neg\varphi(a, b)\}| < k$.*

Remark 6.4 *Assume T has* NIP*, let I be a linearly ordered set without last element, and let $a = (a_i : i \in I)$ be A-indiscernible. Then $\mathrm{Av}(a/\mathfrak{C})$ is finitely satisfiable in $\{a_i : i \in I\}$ and hence it does not fork over $\{a_i : i \in I\}$. Moreover $a_i \models \mathrm{Av}(a/Aa_{<i})$ for all $i \in I$.*

Remark 6.5 *Assume T has* NIP*, let I be a linearly ordered set without last element, and let $a = (a_i : i \in I)$ be A-indiscernible. Then $b \models \mathrm{Av}(a/Aa)$ if and only if $a^\frown(b)$ is A-indiscernible.*

7 Splitting

Definition 7.1 (Reminding) Let $A \subseteq B$, and let $p(x) \in S(B)$. We say that *p splits over A* if for some formula $\varphi(x, y) \in L$ there are tuples $a, b \in B$ such that $a \equiv_A b$, $\varphi(x, a) \in p$, and $\neg\varphi(x,b) \in p$. The definition applies also to $B = \mathfrak{C}$. Note that if $\mathfrak{p} \in S(\mathfrak{C})$, then $\mathfrak{p}$ does not split over A if and only if $\mathfrak{p}^f = \mathfrak{p}$ for all automorphisms $f \in \mathrm{Aut}(\mathfrak{C}/A)$, that is, if and only if $\mathfrak{p}$ is A-*invariant*.

Remark 7.2 *(Reminding) If $p(x) \in S_n(A)$, the number of global extensions of p that do not split over A is at most $2^{(2^{|T|+|A|})}$.*

Remark 7.3 *Let $A \subseteq B$, assume $p(x) \in S(B)$ does not split over A, I is a totally ordered set, and $(a_i : i \in I)$ is a sequence of tuples $a_i \in B$ such that $a_i \models p \restriction Aa_{<i}$. Then $(a_i : i \in I)$ is A-indiscernible.*

Proof: By induction on n it is easy to see that if $i_1 < \ldots < i_n$ and $j_i < \ldots < j_n$, then $a_{i_1}, \ldots, a_{i_n} \equiv_A a_{j_i}, \ldots, a_{j_n}$. □

Lemma 7.4 *Assume T has* NIP*. Let $\mathfrak{p}_1, \mathfrak{p}_2 \in S(\mathfrak{C})$ be global types that do not split over A and assume $\mathfrak{p}_1 \restriction A = \mathfrak{p}_2 \restriction A$. If there is a sequence $(a_i : i < \omega)$ such that $a_i \models \mathfrak{p}_1 \restriction Aa_{<i}$ and $a_i \models \mathfrak{p}_2 \restriction Aa_{<i}$ for all $i < \omega$, then $\mathfrak{p}_1 = \mathfrak{p}_2$.*

Proof: Assume $\varphi(x,y) \in L$ and $\varphi(x,b) \in \mathfrak{p}_1$ and let us check that $\varphi(x,b) \in \mathfrak{p}_2$. Let $(c_i : i < \omega)$ be chosen in such a way that $c_{2\cdot i} \models \mathfrak{p}_1 \restriction Abc_{<2\cdot i}$ and $c_{2\cdot i+1} \models \mathfrak{p}_2 \restriction Abc_{<2\cdot i+1}$. We claim that $(c_i : i < \omega) \equiv_A (a_i : i < \omega)$. Assume, inductively, that $c_0, \ldots, c_{2\cdot n} \equiv_A a_0, \ldots, a_{2\cdot n}$ and suppose $\models \psi(c_0, \ldots, c_{2\cdot n+1})$ where $\psi(x_0, \ldots, x_{2\cdot n+1}) \in L(A)$. Then $\psi(c_0, \ldots, c_{2\cdot n}, x) \in \mathfrak{p}_2$ and by non-splitting over A, $\psi(a_0, \ldots, a_{2\cdot n}, x) \in \mathfrak{p}_2$. Hence $\models \psi(a_0, \ldots, a_{2\cdot n}, a_{2\cdot n+1})$. The odd case is analogous. By the claim, $(c_i : i < \omega)$ is A-indiscernible. Since T has NIP, and $\models \varphi(c_{2\cdot i}, b)$ for all $i < \omega$, also $\models \varphi(c_{2\cdot i+1}, b)$ for all $i < \omega$. Hence $\varphi(x,b) \in \mathfrak{p}_2$. □

Proposition 7.5 *Assume T has* NIP*. Let $\mathfrak{p}_1, \mathfrak{p}_2 \in S(\mathfrak{C})$ be global types that do not split over A and assume $\mathfrak{p}_1 \restriction A = \mathfrak{p}_2 \restriction A$. If there are sequences $(a_i : i < \omega)$, $(b_i : i < \omega)$ such that $a_i \models \mathfrak{p}_1 \restriction Aa_{<i}$ and $b_i \models \mathfrak{p}_2 \restriction Aa_{<i}$ for all $i < \omega$, and moreover $(a_i : i < \omega) \equiv_A (b_i : i < \omega)$, then $\mathfrak{p}_1 = \mathfrak{p}_2$.*

Proof: Let $f \in \mathrm{Aut}(\mathfrak{C}/A)$ be such that $f(a_i : i < \omega) = (b_i : i < \omega)$. Since $\mathfrak{p}_1$ does not split over A, $\mathfrak{p}_1^f = \mathfrak{p}_1$. But then $b_i \models \mathfrak{p}_1 \restriction Ab_{<i}$ and by Lemma 7.4, $\mathfrak{p}_1 = \mathfrak{p}_2$. □

Corollary 7.6 *Assume T has* NIP*. If $p(x) \in S_n(A)$, the number of global extensions of p that do not split over A is $\leq 2^{|A|+|T|}$.*

Proof: By Proposition 7.5 each global non-splitting extension of p is determined by the type over A of an ω-sequence of realizations of p. □

Corollary 7.7 *Assume T has* NIP*. Let $\mathfrak{p} \in S(\mathfrak{C})$ a global type that does not split over A and assume $a_i \models \mathfrak{p} \restriction Aa_{<i}$ for all $i < \omega$. If $\mathrm{Av}((a_i : i < \omega)/\mathfrak{C})$ does not split over A, then $\mathfrak{p} = \mathrm{Av}((a_i : i < \omega)/\mathfrak{C})$.*

Proof: By Lemma 7.4 and Remark 6.4. □

8 Coheirs

Definition 8.1 Let $M \subseteq A$, let $p(x) \in S(M)$ and let $p(x) \subseteq q(x) \in S(A)$. We say that q *is a coheir of* p if it is finitely satisfiable in M. The definition applies also to $A = \mathfrak{C}$. Coheirs are a particular case of non-splitting extensions.

Proposition 8.2 *1. If T has* IP*, then for each $\lambda \geq |T|$ there is some model M of cardinality λ and some $p(x) \in S_1(M)$ having $2^{(2^\lambda)}$ coheirs over $\mathfrak{C}$.*

2. If T has NIP*, then for each $\lambda \geq |T|$, for each model M of cardinality λ, for each $p(x) \in S_n(M)$, $p(x)$ has at most 2^λ coheirs over $\mathfrak{C}$.*

Proof: *1.* By IP, there are $\varphi(x,y) \in L$, $(a_i : i < \lambda)$, and $(b_X : X \subseteq \lambda)$ such that $\models \varphi(a_i, b_X) \Leftrightarrow i \in X$. Let $M \supseteq \{a_i : i < \lambda\}$ be a model of cardinality λ and for each ultrafilter U over λ, let $p_U := \lim_U((a_i : i < \lambda)/\mathfrak{C})$. If $U \neq V$, then there is some $X \in U \smallsetminus V$, which implies $\varphi(x, b_X) \in p_U$ and $\neg\varphi(x, b_X) \in p_V$, and hence $p_U \neq p_V$. Every p_U is finitely satisfiable in M and therefore it is a coheir of $P_U \upharpoonright M$. There are 2^{2^λ} ultrafilters over U and there are only 2^λ complete 1-types over M. Hence, for some $p(x) \in S_1(M)$ there are $2^{(2^\lambda)}$ types p_U extending p.

2. It follows from 7.6 since every coheir is a non-splitting extension. □

9 Forking and Lascar splitting

Definition 9.1 (Reminding) A partial type $\pi(x,a)$ (where $\pi(x,y)$ is over the empty set) *divides over A* if for some A-indiscernible sequence $(a_i : i < \omega)$ of tuples $a_i \equiv_A a$, $\bigcup_{i<\omega} \pi(x, a_i)$ is inconsistent. The type $\pi(x,a)$ *forks over A* if $\pi(x,a)$ implies a disjunction $\varphi_1(x,a_1) \vee \ldots \vee \varphi_n(x,a_n)$ where every $\varphi_i(x,a_i)$ divides over A. For a global type, forking and dividing are always equivalent.

Definition 9.2 (Reminding) The group $\mathrm{Autf}(\mathfrak{C}/A)$ of *strong automorphisms over A* is the subgroup of $\mathrm{Aut}(\mathfrak{C}/A)$ generated by all automorphisms fixing some model containing A. Two tuples a, b have same *Lascar strong type over A*, written $a \overset{\mathrm{Ls}}{\equiv}_A b$, if they are in the same orbit under $\mathrm{Autf}(\mathfrak{C}/A)$. The relation of being two members of an infinite A-indiscernible sequence is type-definable over A by a type $\mathrm{nc}_A(x,y)$. Remind that equality of Lascar strong type over A is the transitive closure of the relation defined by $\mathrm{nc}_A(x,y)$.

Remark 9.3 *If $(a_i : i \in I)$ is an infinite A-indiscernible sequence, then for any $i_0 < \ldots < i_n \in I$, and $j_0 < \ldots < j_n \in I$,*

$$a_{i_0} \ldots a_{i_n} \overset{\mathrm{Ls}}{\equiv}_A a_{j_0} \ldots a_{j_n}$$

Proof: One can assume $i_n < j_0$ and this case is easy since $\models \mathrm{nc}_A(a_{i_0} \ldots a_{i_n}; a_{j_0} \ldots a_{j_n})$. □

Definition 9.4 Let $A \subseteq B$, and let $p(x) \in S(B)$. We say that *p strongly splits over A* if for some $\varphi(x,y) \in L$ there are tuples $a, b \in B$ such that $\models \mathrm{nc}_A(a,b)$, $\varphi(x,a) \in p$, and $\neg\varphi(x,b) \in p$. We say that *p Lascar-splits over A* if for some formula $\varphi(x,y) \in L$ there are tuples $a, b \in B$ such that $a \overset{\mathrm{Ls}}{\equiv}_A b$, $\varphi(x,a) \in p$, and $\neg\varphi(x,b) \in p$. If A is a model, Lascar-splitting over A is equivalent to splitting over A. These definitions apply also to $B = \mathfrak{C}$. Note that if $\mathfrak{p} \in S(\mathfrak{C})$, then $\mathfrak{p}$ does not Lascar-split over A if and only if $\mathfrak{p}^f = \mathfrak{p}$ for all strong automorphisms $f \in \mathrm{Autf}(\mathfrak{C}/A)$.

Remark 9.5 *Let $A \subseteq B$, and let $p(x) \in S(B)$.*

1. *If $p(x)$ strongly splits over A, then $p(x)$ Lascar-splits over A.*

2. If $p(x)$ Lascar-splits over A, then p splits over A.

Proposition 9.6 *Assume T has* NIP*. A global type $\mathfrak{p} \in S(\mathfrak{C})$ does not fork over A if and only if it does not Lascar-split over A.*

Proof: Assume first $\mathfrak{p}$ does not fork over A. Let $\varphi(x, y) \in L$. It is enough to check that $\varphi(x, b) \in \mathfrak{p}$ whenever $\varphi(x, a) \in \mathfrak{p}$ and $\models \mathrm{nc}_A(a, b)$. Let $(a_i : i < \omega)$ be some A-indiscernible sequence with $a = a_0$ and $b = a_1$. If $\varphi(x, a) \in \mathfrak{p}$ but $\varphi(x, b) \notin \mathfrak{p}$ then since $\mathfrak{p}$ does not divide over A and the sequence $(a_{2\cdot i}a_{2\cdot i+1} : i < \omega)$ is A-indiscernible, it follows that $\{\varphi(x, a_{2\cdot i}) \wedge \neg\varphi(x, a_{2\cdot i+1}) : i < \omega\}$ is consistent, which implies that $\mathrm{alt}(\varphi(x, y)) = \infty$.

Assume now $\mathfrak{p}$ does not Lascar-split over A. We will check that no formula in $\mathfrak{p}$ divides over A. Let $\varphi(x, y) \in L$ and assume that $\varphi(x, a) \in \mathfrak{p}$ and $(a_i : i < \omega)$ is an A-indiscernible sequence with $a = a_0$. Then $a_i \overset{\mathrm{Ls}}{\equiv}_A a$ and therefore $\varphi(x, a_i) \in \mathfrak{p}$ for all $i < \omega$. This shows that $\{\varphi(x, a_i) : i < \omega\}$ is consistent. □

Definition 9.7 Extending slightly terminology of [11], we say that B *is complete over* A if $A \subseteq B$, and every n-type over A is realized in B. We also say that B *is ω-saturated over* A if $A \subseteq B$ and for each finite $B_0 \subseteq B$ every n-type over AB_0 is realized in B. This last condition implies that B is an ω-saturated model. Obviously, if M is ω-saturated over A, then M is complete over A. In particular, the monster model $\mathfrak{C}$ is complete over any set A. This notions can also be extended to Lascar strong types. For instance, we say that B *is Lascar-complete over* A if $A \subseteq B$, and every finitary Lascar strong type over A is realized in B.

Remark 9.8 *If M is ω-saturated over A and $p(x) \in S(M)$, then*

1. *p forks over A if and only if p divides over A.*
2. *p strongly splits over A if and only if p Lascar-splits over A.*

Remark 9.9 *If B is A-complete and $p(x) \in S(B)$ forks over A, then p splits over A or p divides over A.*

Proof: Assume $p(x)$ forks over A but does not split over A. There are some formulas $\theta(x, z), \varphi_1(x, y_1), \ldots, \varphi_n(x, y_n) \in L$, some tuple $c \in B$ and some tuples $a_1, \ldots, a_n$ such that $\theta(x, c) \in p(x)$, $\theta(x, c) \vdash \varphi_1(x, a_1) \vee \ldots \vee \varphi_n(x, a_n)$, and each $\varphi(x, a_i)$ divides over A. By A-completeness of B we can choose $d, b_1, \ldots, d_n \in B$ such that $ca_1 \ldots a_n \equiv_A db_1 \ldots b_n$. Then $\theta(x, d) \vdash \varphi_1(x, b_1) \vee \ldots \vee \varphi_n(x, b_n)$ and $\theta(x, d) \in p(x)$. Hence $\varphi_i(x, b_i) \in p$ for some i. Since $\varphi_i(x, b_i)$ divides over A, $p(x)$ divides over A. □

Remark 9.10 *A careful reading of the proof of Proposition 9.6 shows that if $A \subseteq B$ and $p(x) \in S(B)$, then:*

1. *If T has* NIP *and p does not divide over A, then p does not strongly split over A.*
2. *If B is ω-saturated over A, and p does not strongly split over A, then p does not divide over A.*

Hence, if B is ω-saturated over A, then over A

$$\textit{forking} = \textit{dividing} \Rightarrow \textit{strongly splitting} = \textit{Lascar splitting} \Rightarrow \textit{splitting}$$

and if moreover T has NIP*, then forking, dividing, strongly splitting, and Lascar splitting over A are all the same. If additionally A is a model, then splitting over A is also equivalent to all these properties.*

Remark 9.11 *Assume T has* NIP*. If $p(x) \in S(B)$ does not fork over $A \subseteq B$, then p does not Lascar-split over A.*

Proof: If p does not fork over A, p has a global extension $\mathfrak{p}$ that does not fork over A. By Proposition 9.6, $\mathfrak{p}$ does not Lascar split over A. Clearly, p does not Lascar-split over A neither. $\square$

Proposition 9.12 *Let M be ω-saturated over $A \subseteq M$. If $a \downarrow_A M$ and $b \downarrow_{Aa} M$, then $ab \downarrow_A M$.*

Proof: A well-known property of dividing is: if $\mathrm{tp}(a/M)$ does not divide over A and $\mathrm{tp}(b/Ma)$ does not divide over Aa, then $\mathrm{tp}(ab/M)$ does not divide over A. Since M is ω-saturated over A forking and dividing over A for types over M are equivalent. $\square$

Corollary 9.13 *If $a \downarrow_A A$ and $b \downarrow_{Aa} A$, then $ab \downarrow_A A$.*

Proof: Choose $M \supseteq A$, a model ω-saturated over A. Let $a' \equiv_A a$ such that $a' \downarrow_A M$. Choose b' such that $ab \equiv_A a'b'$. Since $b' \downarrow_{Aa'} A$, we may choose b'' such that $b'' \downarrow_{Aa'} M$ and $b'' \equiv_{Aa'} b'$. By Proposition 9.12, $a'b'' \downarrow_A M$. In particular $a'b'' \downarrow_A A$. Since $ab \equiv_A a'b''$, $ab \downarrow_A A$. $\square$

10 Nonsplitting extensions and products

Proposition 10.1 *Let B be complete over A and $p(x) \in S(B)$. If p does not split over A, then for every $C \supseteq B$ there is a unique $q(x) \in S(C)$ extending p that does not split over A. Similarly for Lascar-splitting if B is Lascar-complete over A.*

Proof: $q(x) := p(x) \cup \bigcup_{\varphi(x,y)\in L}\{\varphi(x,a) : a \in C \text{ and } \varphi(x,a') \in p \text{ for some } a' \equiv_A a\}$ and write $a' \overset{\mathrm{Ls}}{\equiv}_A a$ in the second case. $\square$

Definition 10.2 Assume B is complete over A and $p(x) \in S(B)$ does not split over A. For any set $C \supseteq B$, $p|_A C$ is the only complete extension of p to C that does not split over A.

Remark 10.3 *Assume B is complete over A and $p(x) \in S(B)$ does not split over A.*

1. *For any $D \supseteq C \supseteq B$, $(p|_A C)|_A D = p|_A D$.*
2. *For any $C \supseteq B$, if the sequence $(a_i : i < \omega)$ is chosen in such a way that $a_i \models p|_A Ca_{<i}$, then $(a_i : i < \omega)$ is C-indiscernible.*

Remark 10.4 *Assume B is AA'-complete and $p(x) \in S(B)$ does not split over A nor over A'. Then for any $C \supseteq B$, $p|_A C = p|_{A'} C$.*

Proof: Assume $\varphi(x,y) \in L$, $a \in C$ and $\varphi(x,a) \in p|_A C$. Choose $b \in B$ such that $a \equiv_{AA'} b$. Since $a \equiv_A b$, $\varphi(x,b) \in p \subseteq p|_{A'} C$. Since $a \equiv_{A'} b$, $\varphi(x,a) \in p|_{A'} C$. $\square$

Definition 10.5 Assume $p(x) \in S(B)$ does not split over some $A \subseteq B$ of cardinality $\kappa \geq \omega$ and B is A'-complete for all $A' \subseteq B$ of cardinality κ. Then for any $C \supseteq B$, $p|C$ is the unique extension of p over C that does not split over a subset of B of cardinality κ. It is independent of the choice of κ.

Definition 10.6 Assume B is complete over A, $p(x), q(y) \in S(B)$ and $q(y)$ does not split over A. We define the *product*[1] $p \otimes_A q$ as $\mathrm{tp}(ab/B)$ where $a \models p$ and $b \models q|_A Ba$. It is independent of the choice of a, b. If $\kappa = |A|$ and B is A'-complete for all $A' \subseteq B$ of cardinality κ (for instance, B is a $\kappa^+ + \omega$-saturated model), then it is also independent of the choice of A (and κ) and we denote the product by $p \otimes q$.

Lemma 10.7 *Assume B is complete over A, and $p(x), q(y) \in S(B)$ do not split over A. Then $p \otimes_A q$ does not split over A.*

Proof: Assume $\varphi(x, y, z) \in L$, $c \in B$, and $\varphi(x, y, c) \in p \otimes_A q$. Choose $a \models p$, choose $b \models q|_A Ba$, and let $c' \in B$ be such that $c \equiv_A c'$. Since p does not split over A, $ac \equiv_A ac'$ and hence $\varphi(a, y, c') \in q|_A Ba$ and $\models \varphi(a, b, c')$. It follows that $\varphi(x, y, c') \in p \otimes_A q$. □

Proposition 10.8 *Assume B is complete over A and $p(x), q(y), r(z) \in S(B)$ do not split over A.*

1. *For any $C \supseteq B$, $(p|_A C \otimes_A q|_A C) = (p \otimes_A q)|_A C$.*
2. *$(p \otimes_A q) \otimes_A r = p \otimes_A (q \otimes_A r)$.*

Proof: *1.* $(p|_A C \otimes_A q|_A C)$ is an extension of $p \otimes_A q$ over C and it does not split over A.

2. Take $a \models p$, $b \models q|_A Ba$ and $c \models r|_A Bab$. Clearly, $abc \models (p \otimes q) \otimes_A r$. On the other hand, $bc \models (q|_A Ba \otimes_A r|_A Ba)$ and by *1* $bc \models (q \otimes_A r)|_A Ba$. Hence $abc \models p \otimes_A (q \otimes_A r)$. □

Definition 10.9 Assume B is complete over A and $p(x) \in S(B)$ does not split over A. The n-th power $p(x_1, \ldots, x_n)^{(n)_A}$ is defined for $n \geq 1$ as the product $p(x_1) \otimes_A \ldots \otimes_A p(x_n)$. By associativity, it is well-defined. It is a complete type over B and it does not split over A. We define the *ω-power of p* as $p^{(\omega)_A}(x_i : i < \omega) = \bigcup_{i<\omega} p^{(i+1)_A}(x_0, \ldots, x_i)$. Again, it is a complete type over B and it does not split over A. If $\kappa = |A|$ and B is A'-complete for all $A' \subseteq B$ of cardinality κ then powers of p are independent of the choice of A and we can write $p^{(n)}$ and $p^{(\omega)}$

Remark 10.10 *Assume B is complete over A and $p(x) \in S(B)$ does not split over A.*

1. *$(a_i : i < \omega) \models p^{(\omega)_A}$ if and only if $a_i \models p|_A Ba_{<i}$ for all $i < \omega$.*
2. *If $(a_i : i < \omega) \models p^{(\omega)_A}$, then $(a_i : i < \omega)$ is indiscernible over B.*
3. *If $(a_i : i < \omega) \models p^{(\omega)_A}$, then $(a_i : j \leq i < \omega) \models p^{(\omega)_A}|_A Ba_{<j}$.*

Proof: *3.* Clear since (by *1*) $p^{(\omega)_A}|Ba_{<j} = (p|Ba_{<j})^{(\omega)_A}$. □

[1] We have changed the order of p and q in the definition of product with the purpose of making the definition of the power easier to understand.

11 Lascar strong types and KP-splitting

The results in Section 10 can be generalized to non Lascar-splitting extensions and Lascar-complete sets. In particular:

Remark 11.1 *Let B be Lascar-complete over A and let $p(x) \in S(B)$.*

1. *If $p(x)$ does not Lascar-split over A, then for any $C \supseteq B$, we also denote by $p|_A C$ the unique extension of p to C that does not Lascar-split over A.*
2. *If $q(y) \in S(B)$ does not Lascar-split over A, we also denote $p(x) \otimes_A q(y)$ the type of a tuple ab such that $a \models p$ and $b \models q|_A Ba$ (in the new sense).*

Proposition 11.2 *Assume T has* NIP *and $B \supseteq A$ is Lascar-complete over A and $p(x) \in S(B)$. Then p does not fork over A if and only if p does not Lascar-split over A.*

Proof: One direction follows from Remark 9.11. Now, if p does not Lascar-split over A, then p has a global extension $\mathfrak{p}$ that does not Lascar-split over A. By Proposition 9.6, $\mathfrak{p}$ does not fork over A. Hence p does not fork over A. □

Proposition 11.3 *Assume T has* NIP.[2] *Let $B \supseteq A$ be Lascar-complete over A. If the types $p(x), q(y) \in S(B)$ do not Lascar-split over A, then $p(x) \otimes_A q(y)$ does not Lascar-split over A.*

Proof: Extend B to an ω-saturated over A model $M \supseteq B$. Assume $p'(x), q'(y) \in S(M)$ do not Lascar-split over A. Let $a \models p'$ and $b \models q'|_A Ma$. Then $p'(x) \otimes_A q'(y) = \mathrm{tp}(ab/M)$. Since $a \mathop{\smile\!\!\!\!\!\mid}_A M$ and $b \mathop{\smile\!\!\!\!\!\mid}_{Aa} M$, by Proposition 9.12, $ab \mathop{\smile\!\!\!\!\!\mid}_A M$. Hence $\mathrm{tp}(ab/M)$ does not Lascar-split over A. Now consider $p(x) \otimes q(y)$ for $p, q \in S(B)$. We have shown that $p|_A M \otimes q|_A M$ does not Lascar-split over A. Since this type extends $p(x) \otimes q(y)$, it follows that $p(x) \otimes q(y)$ does not Lascar-split over A. □

Proposition 11.4 *Assume T has* NIP*. Assume $B \supseteq A$ is Lascar-complete over A and $p(x), q(y), r(z) \in S(B)$ do not Lascar split over A.*

1. *For any $C \supseteq B$, $(p|_A C \otimes_A q|_A C) = (p \otimes_A q)|_A C$.*
2. *$(p \otimes_A q) \otimes_A r = p \otimes_A (q \otimes_A r)$.*

Proof: Like the proof of proposition 10.8, but using now Proposition 11.3. □

Remark 11.5 *Assume T has* NIP*. Let $B \supseteq A$ be Lascar-complete over A. Assuming $p(x) \in S(B)$ does not Lascar-split over A, the powers $p^{(n)_A}$ and $p^{(\omega)_A}$ are defined in analogous way as we did in the nonsplitting case, using associativity of the product. It follows from Proposition 11.3 that the powers $p^{(n)_A}$ and $p^{(\omega)_A}$ do not Lascar-split over A.*

Lemma 11.6 *Assume T has* NIP*. Let $B \supseteq A$ be Lascar-complete over A, and assume $p(x) \in S(B)$ does not Lascar-split over A.*

1. *$(a_i : i < \omega) \models p^{(\omega)_A}$ if and only if $a_i \models p|_A Ba_{<i}$ for all $i < \omega$.*

[2] In fact the assumption of NIP is unnecessary since one can use left transitivity of $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{i}}$ (see Section 18). The same applies to Proposition 11.4, Remark 11.5 and Lemma 11.6.

2. *If* $a, b \models p$ *then* $a \overset{\text{Ls}}{\equiv}_A b$.

3. *If* $a_1 \dots a_n \models p^{(n)_A}$ *and* $b_1 \dots b_n \models p^{(n)_A}$, *then* $a_1 \dots a_n \overset{\text{Ls}}{\equiv}_A b_1 \dots b_n$.

4. *If* $(a_i : i < \omega) \models p^{(\omega)_A} \restriction A$, *then* $(a_i : i < \omega)$ *is indiscernible over* A.[3]

5. *If* $(a_i : i < \omega) \models p^{(\omega)_A}$, *then* $(a_i : j \leq i < \omega) \models p^{(\omega)_A}|_A a_{<j}$.

Proof: *1.* Clear.

2. Choose $a' \in B$ such that $a \overset{\text{Ls}}{\equiv}_A a'$ and then choose n such that $\models \text{nc}^n_A(a, a')$. Then $\models \text{nc}^n_A(b, a')$ and hence $b \overset{\text{Ls}}{\equiv}_A a' \overset{\text{Ls}}{\equiv}_A a$.

3. By *2*, since $p^{(n)_A}$ does not Lascar-split over A.

4. We may assume $(a_i : i < \omega) \models p^{(\omega)_A}$ and then we use *3* since $a_{i_1}, \dots, a_{i_n} \models p^{(n)_A}$ whenever $i_0 < \dots < i_n$.

5. Like in Remark 10.10. □

Proposition 11.7 *Let* $p(x) \in S(A)$ *and assume there is a global extension* $\mathfrak{p} \in S(\mathfrak{C})$ *of* p *that does not Lascar split over* A. *For any* $c, d \models p$ *the following are equivalent:*

1. $c \overset{\text{Ls}}{\equiv}_A d$

2. *For some Lascar* A*-complete* B *there is a non Lascar-splitting extension* $q(x) \in S(B)$ *of* p *and some* $(a_i : i < \omega)$ *such that both* $c^\frown(a_i : i < \omega)$ *and* $d^\frown(a_i : i < \omega)$ *realize* $q^{(\omega)_A} \restriction A$.

3. $\models \text{nc}^2_A(c, d)$

Proof: *1* ⇒ *2*. Choose $B \supseteq A$ Lascar complete over A. By point 1 of Lemma 11.6, $a \overset{\text{Ls}}{\equiv}_A b$ whenever $a, b \models \mathfrak{p} \restriction B$. We can assume $c \models \mathfrak{p} \restriction B$ and hence $\mathfrak{p} \restriction B \vdash \text{Lstp}(c/A)$ (otherwise we choose $a \models \mathfrak{p} \restriction B$ and some $f \in \text{Aut}(\mathfrak{C}/A)$ such that $f(a) = c$, and we replace $\mathfrak{p}$ and B by $\mathfrak{p}^f$ and $f(B)$). Now let $q = \mathfrak{p} \restriction Bcd$, let $(a_i : i < \omega)$ be a realization of the power $q^{(\omega)_A}$ and let $\bar{a} = (a_i : 0 < i < \omega)$. Since $c \overset{\text{Ls}}{\equiv}_A d$ and $\text{tp}(\bar{a}/Bcd) = \text{tp}((a_i : i < \omega)/Bcd)$ does not Lascar-split over A, $c\bar{a} \equiv_A d\bar{a}$. By point 1 of Lemma 11.6, $\bar{a} \models q^{(\omega)_A}|Bcda_0$ (a type that does not Lascar-split over A) and therefore $a_0 \overset{\text{Ls}}{\equiv}_A c$ implies $a_0\bar{a} \equiv_A c\bar{a}$. Since $a_0\bar{a} \models q^{(\omega)_A}$ we conclude that $c\bar{a}$ and $d\bar{a}$ realize $q^{(\omega)_A} \restriction A$.

2 ⇒ *3*. Clear since (By point 4 Lemma 11.6) any realization of $q^{(\omega)_A} \restriction A$ is A-indiscernible.

3 ⇒ *1* is obvious. □

Definition 11.8 (Reminding) For any given length of tuples, for any set A, there is a least bounded type-definable over A equivalence relation E_{KP_A}, the Kim-Pillay relation. It is refined by E_{L_A}, the Lascar relation, which is the least bounded A-invariant equivalence relation. If E_{L_A} is type-definable, then $E_{L_A} = E_{KP_A}$. In any case, E_{L_A} is equality of Lascar strong type over A. Similarly, E_{KP_A} is equality of KP-type over A. We write $E_{KP_A}(a, b) \Leftrightarrow a \overset{\text{KP}}{\equiv}_A b$. The KP-type over A of a tuple a is $\text{tp}(a/\text{bdd}(A))$, where $\text{bdd}(A)$ is the class of all hyperimaginaries that have a bounded A-orbit, that is, $a \overset{\text{KP}}{\equiv}_A b$ iff $E(a, b)$ for all bounded A-type-definable equivalence relation E iff $\text{tp}(a/\text{bdd}(A)) = \text{tp}(b/\text{bdd}(A))$.

[3]This can be proved directly if one defines $p^{(n)_A}$ by left nesting, since this would be the type of each subtuple of length n.

Corollary 11.9 *Assume T has* NIP *and $p(x) \in S(A)$ does not fork over A. For any $c, d \models p$,*

$$c \overset{\text{Ls}}{\equiv}_A d \text{ if and only if } \models \mathrm{nc}^2_A(c,d)$$

Hence, in p, equality of Lascar strong type over A, $\overset{\text{Ls}}{\equiv}_A$, is type definable over A and coincides with $\overset{\text{KP}}{\equiv}_A$, equality of KP-type over A.

Proof: If p does not fork over A, p has a global nonforking extension $\mathfrak{p}$. By Proposition 9.6, $\mathfrak{p}$ does not Lascar-split over A. The rest follows from Proposition 11.7. □

Proposition 11.10 *Assume T has* NIP. *Let $\mathfrak{p}_1, \mathfrak{p}_2 \in S(\mathfrak{C})$ be global types that do not Lascar-split over A and assume $\mathfrak{p}_1 \upharpoonright A = \mathfrak{p}_2 \upharpoonright A$. Let $B \supseteq A$ be Lascar-complete over A, let $p = \mathfrak{p}_1 \upharpoonright B$ and let $a = (a_i : i < \omega) \models p^{(\omega)_A}$. If $\mathfrak{p}_1 \upharpoonright Aa = \mathfrak{p}_2 \upharpoonright Aa$, then $\mathfrak{p}_1 = \mathfrak{p}_2$.*

Proof: We claim that if $a' = (a_i : i < \alpha)$ is an A-indiscernible sequence extending a, then $a'c$ is also A-indiscernible for any $c \models \mathfrak{p}_1 \upharpoonright Aa'$ or $c \models \mathfrak{p}_2 \upharpoonright Aa'$. Consider the case $c \models \mathfrak{p}_1 \upharpoonright Aa'$. Assume $i_0 < \ldots < i_n < \alpha$, $\psi(x_0, \ldots, x_n, y) \in L(A)$ and $\models \psi(a_{i_0}, \ldots, a_{i_n}, c)$. Then $\psi(a_{i_0}, \ldots, a_{i_n}, y) \in \mathfrak{p}_1$. Since $\mathfrak{p}_1$ does not Lascar-split over A and $a_{i_0} \ldots a_{i_n} \overset{\text{Ls}}{\equiv}_A a_0 \ldots a_n$, $\psi(a_0, \ldots, a_n, y) \in \mathfrak{p}_1$. Since $a_{n+1} \models p|_A Ba_0 \ldots a_n = \mathfrak{p}_1 \upharpoonright Ba_0 \ldots a_n$, $\models \psi(a_0, \ldots, a_n, a_{n+1})$. The case $c \models \mathfrak{p}_1 \upharpoonright Aa'$ is similar but one one needs the assumption $\mathfrak{p}_1 \upharpoonright Aa = \mathfrak{p}_2 \upharpoonright Aa$.

Now assume $\varphi(x, y) \in L$, $\varphi(x, b) \in \mathfrak{p}_1$ and $\neg\varphi(x, b) \in \mathfrak{p}_2$. Construct $(c_i : i < \omega)$ in such a way that $c_{2\cdot i} \models \mathfrak{p}_1 \upharpoonright Aabc_{<2\cdot i}$ and $c_{2\cdot i+1} \models \mathfrak{p}_2 \upharpoonright Aabc_{<2\cdot i+1}$. Note that a is A-indiscernible. By the claim $a^\frown(c_i : i < \omega)$ is also A-indiscernible. Since $\models \varphi(a_{2\cdot i}, b)$ and $\models \varphi(a_{2\cdot i+1}, b)$ we see that $\mathrm{alt}(\varphi) = \infty$, contradicting NIP of T. □

Definition 11.11 We say that $p(x) \in S(B)$ KP-splits over $A \subseteq B$ if there are tuples $a, b \in B$ and $\varphi(x, y) \in L$ such that $\varphi(x, a) \in p$, $\neg\varphi(x, b) \in p$, and $a \overset{\text{KP}}{\equiv}_A b$. Note that Lascar-splitting implies KP-splitting and KP-splitting implies splitting. Note also that a global type $\mathfrak{p}$ does not KP-split over A if and only if it is $\mathrm{bdd}(A)$-invariant, that is, $\mathfrak{p}^f = \mathfrak{p}$ for all $f \in \mathrm{Aut}(\mathfrak{C}/\mathrm{bdd}(A))$.

Lemma 11.12 *Assume T has* NIP. *Let $f \in \mathrm{Aut}(\mathfrak{C}/A)$ an let $\mathfrak{p}$ be a global type that does not Lascar-split over A. If for each $n < \omega$, for each $a \models \mathfrak{p}^{(n)} \upharpoonright A$, $a \overset{\text{Ls}}{\equiv}_A f(a)$, then $\mathfrak{p}^f = \mathfrak{p}$.*

Proof: Clearly, $\mathfrak{p} \upharpoonright A = \mathfrak{p}^f \upharpoonright A$. Choose B Lascar-complete over A and let $p = \mathfrak{p} \upharpoonright B$ and $a = (a_i : i < \omega) \models p^{(\omega)_A}$. By Proposition 11.10 it will suffice to prove $\mathfrak{p} \upharpoonright Aa = \mathfrak{p}^f \upharpoonright Aa$. By Corollary 11.9, if $a_{<n} \overset{\text{Ls}}{\equiv}_A f(a_{<n})$ for all $n < \omega$, then $a \overset{\text{Ls}}{\equiv}_A f(a)$. Let $\varphi(x, y) \in L(A)$ and assume $\varphi(x, a) \in \mathfrak{p}$. Since $\mathfrak{p}$ does not Lascar-split over A and $a \overset{\text{Ls}}{\equiv}_A f^{-1}(a)$, $\varphi(x, f^{-1}(a)) \in \mathfrak{p}$. Then $\varphi(x, a) \in \mathfrak{p}^f$. It follows that $\mathfrak{p} \upharpoonright Aa = \mathfrak{p}^f \upharpoonright Aa$. □

Proposition 11.13 *Assume T has* NIP. *Let B be Lascar-complete over A and $p(x) \in S(B)$. Then p Lascar-splits over A if and only if p KP-splits over A.*

Proof: It is enough to check that a type does not KP-split if it does not Lascar-split and it is enough to consider the case of a global type $\mathfrak{p}$. Assume $\mathfrak{p}$ does not Lascar-split over A, and let $f \in \mathrm{Aut}(\mathfrak{C}/\mathrm{bdd}(A))$. We can check that $\mathfrak{p}^f = \mathfrak{p}$ using Lemma 11.12. since $\mathfrak{p}^{(n)_A}$ does not Lascar-split over A and a and $f(a)$ are realizations of $\mathfrak{p}^{(n)_A} \upharpoonright A$ such that $a \overset{\text{KP}}{\equiv}_A f(a)$, and then, by Proposition 11.7, $a \overset{\text{Ls}}{\equiv}_A f(a)$. □

12 Morley sequences

Definition 12.1 Assume the index set I is linearly ordered by $<$. We say that $(a_i : i \in I)$ is a *Morley sequence over A* if it is A-indiscernible and it is A-independent in the sense of forking: for all $i \in I$, $\mathrm{tp}(a_i/Aa_{<i})$ does not fork over A. If $p(x)$ is the common type $\mathrm{tp}(a_i/A)$ of all a_i, we say that $(a_i : i \in I)$ is a Morley sequence in p.

Remark 12.2 *If B is complete over A, and $p(x) \in S(B)$ does not split over A, then any $(a_i : i < \omega) \models p^{(\omega)_A} \restriction A$ is a Morley sequence in $p \restriction A$.*

Lemma 12.3 *Assume T has* NIP. *Let $\mathfrak{p}_1, \mathfrak{p}_2$ be global types that do not fork over A. Let I be infinite and linearly ordered by $<$. Let $a = (a_i : i \in I)$ be a Morley sequence in $p(x) = \mathfrak{p}_1 \restriction A = \mathfrak{p}_2 \restriction A$. If $a_i \models \mathfrak{p}_1 \restriction Aa_{<i} = \mathfrak{p}_2 \restriction Aa_{<i}$ for all $i \in I$, then $\mathfrak{p}_1 = \mathfrak{p}_2$.*

Proof: Note first that, by compactness, we can assume that $I = \omega$ with its standard ordering. The rest is like the proof of Proposition 11.10. □

Proposition 12.4 *Assume T has* NIP *and I is a linearly ordered set without last element.*

1. *If $a = (a_i : i \in I)$ is a Morley sequence in $p(x) \in S(A)$, then there is a unique global type $\mathfrak{p} \supseteq p$ such that $\mathfrak{p}$ does not fork over A and $a_i \models \mathfrak{p} \restriction Aa_{<i}$ for all $i \in I$. Moreover $\mathfrak{p} \restriction Aa = \mathrm{Av}(a/Aa)$.*
2. *If $\mathfrak{p} \in S(\mathfrak{C})$ does not fork over A, there is a Morley sequence $(a_i : i < \omega)$ in $p = \mathfrak{p} \restriction A$ whose associated global type as in the previous point is $\mathfrak{p}$.*

Proof: *1.* If $p_i(x) = \mathrm{tp}(a_i/Aa_{<i})$, then $\bigcup_{i\in I} p_i(x) = \mathrm{Av}(a/Aa)$ does not fork over A and therefore it has a global nonforking (over A) extension $\mathfrak{p}(x)$. Then $\mathfrak{p}(x)$ does not Lascar-split over A and $a_i \models \mathfrak{p} \restriction Aa_{<i}$ for all $i \in I$. Uniqueness follows from Lemma 12.3.

2. Let $M \supseteq A$ be a model. Then $\mathfrak{p}$ does not split over M. Choose $(a_i : i < \omega)$ in such a way that $a_i \models \mathfrak{p} \restriction Ma_{<i}$. Then $(a_i : i < \omega)$ is M-indiscernible and hence A-indiscernible. It is therefore a Morley sequence in $\mathfrak{p} \restriction A$. □

Proposition 12.5 *Assume I is a linearly ordered set. If $a = (a_i : i \in I)$ is A-independent, then $a \mathop{\smile\!\!\!\!\!\!\;\;|}_A A$.*

Proof: We may assume I is finite. The we can proceed by induction in $|I|$ using Corollary 9.13. □

Proposition 12.6 *Assume T has* NIP *and $\mathfrak{p}$ does not split over A. The sequence $a = (a_i : i < \omega)$ is a Morley sequence over A with global type $\mathfrak{p}$ if and only if $a \models \mathfrak{p}^{(\omega)} \restriction A$.*

Proof: Let $M \supseteq A$ be Lascar-complete over A and let $p = \mathfrak{p} \restriction M$. Note that $p^{(\omega)_A}|\mathfrak{C} = (p|\mathfrak{C})^{(\omega)_A} = \mathfrak{p}^{(\omega)}$ and hence $p^{(\omega)_A} \restriction A = \mathfrak{p}^{\omega} \restriction A$. Now, assume $a = (a_i : i < \omega)$ is a Morley sequence over A with global type $\mathfrak{p}$ and let $b = (b_i : i < \omega) \models p^{(\omega)_A}$. By induction it is easy to see that $a_{<i} \equiv_A b_{<i}$ for all $i < \omega$. The other direction follows from Remark 12.2. □

Question 12.7 *Does Proposition 12.6 hold assuming only that $\mathfrak{p}$ does not fork over A ? The problem is with the direction from left to right.*

13 Special sequences and eventual types

Definition 13.1 An infinite sequence $a = (a_i : i \in I)$ is A-*special* if it is A-indiscernible and every $b = (b_i : i \in I) \equiv_A (a_i : i \in I)$ can be extended to an A-indiscernible sequence $b^\frown(c)$ by adding a new tuple c such that also $a^\frown(c)$ is A-indiscernible.

Lemma 13.2 *Let a, b be infinite A-indiscernible sequences and assume a and b have the same Ehrenfeucht-Mostowski set over A. Then a is A-special if and only if b is A-special.*

Proof: Let $\Phi(x_i : i < \omega)$ be the Ehrenfeucht-Mostowski set over A of the A-indiscernible sequence $a = (a_i : i \in I)$. By definition, for any $\varphi(x_1, \ldots, x_n) \in L(A)$, $\varphi \in \Phi$ if and only if $\models \varphi(a_{i_1}, \ldots, a_{i_n})$ for all (for some) $i_1 < \ldots < i_n$ in I. Assume $b = (b_j : j \in J)$ is A-indiscernible, with the same Ehrenfeucht-Mostowski set, and assume $b \equiv_A b' = (b'_j : j \in J)$. Let $\varphi(x_1, \ldots, x_n, x_{n+1}) \in \Phi$ and $j_1 < \ldots < j_n$ in J. It is enough to check that $\varphi(b_{j_1}, \ldots, b_{j_n}, x_{n+1}) \wedge \varphi(b'_{j_1}, \ldots, b'_{j_n}, x_{n+1})$ is consistent. Let $p(x_1, \ldots, x_n) = \mathrm{tp}(b_{j_1}, \ldots, b_{j_n}/A)$ and choose $i_1 < \ldots < i_n$ in I. Since a is A-special and also $p = \mathrm{tp}(a_{i_1}, \ldots, a_{i_n}/A)$, $p(x_1, \ldots, x_n) \vdash \exists x_{n+1}(\varphi(a_{i_1}, \ldots, a_{i_n}, x_{n+1}) \wedge \varphi(x_1, \ldots, x_n, x_{n+1}))$. It follows that $p(x_1, \ldots, x_n) \vdash \exists x_{n+1}(\varphi(b_{j_1}, \ldots, b_{j_n}, x_{n+1}) \wedge \varphi(x_1, \ldots, x_n, x_{n+1}))$ and therefore $\models \exists x_{n+1}(\varphi(b_{j_1}, \ldots, b_{j_n}, x_{n+1}) \wedge \varphi(b'_{j_1}, \ldots, b'_{j_n}, x_{n+1}))$ □

Lemma 13.3 *Assume T has* NIP *and a is A-special. Let $n < \omega$ and suppose $a_i \equiv_A a$ for all $i < n$. Then for some tuple b all the sequences $a_i{}^\frown(b)$ are A-indiscernible.*

Proof: Since a is A-special, we can construct a sequence $d = (d_i : i < \omega)$ such that $a^\frown d$ is A-indiscernible and for each $i < n$, if $d^i = (d_{j \cdot n+i} : j < \omega)$, then $a_i^\frown d^i$ is A-indiscernible. Now if $b \models \mathrm{Av}(d/A(a_i : i < n)d) = \mathrm{Av}(d^i/A(a_i : i < n)d)$, then $a_i^\frown(b)$ is A-indiscernible for all $i < n$. □

Proposition 13.4 *Assume T has* NIP. *If $a = (a_i : i \in I)$ is A-special then for any family $(b^i : i < \lambda)$ where $b^i \equiv_A a$, for any linearly ordered set J there is some sequence $c = (c_j : j \in J)$ such that every $b^{i\frown} c$ is A-indiscernible.*

Proof: Notice that if a is A-special and we extend it to an A-indiscernible sequence adding finitely many tuples $c_1, \ldots, c_n$ at the end of a, then the extended sequence $a^\frown(c_1, \ldots, c_n)$ has the same Ehrenfeucht-Mostowski set over A and it is therefore A-special. Using this and compactness, it is easily seen that it suffices to apply Lemma 13.3. □

Definition 13.5 Assume I is an infinite linearly ordered set. Let $a = (a_i : i \in I)$ be A-special. The *eventual type of a over $B \supseteq A$*, $\mathrm{Ev}_A(a/B)$, is the set of formulas $\varphi(x) \in L(B)$ such that for any $b \equiv_A a$ there is some ω-sequence c such that $b^\frown c$ is A-indiscernible and $\varphi(x) \in \mathrm{Av}(b^\frown c/B)$. Usually A is clear from the context, and we can omit it and write $\mathrm{Ev}(a/B)$.

Remark 13.6 *If a is A-special, and $C \supseteq B \supseteq A$, then* $\mathrm{Ev}(a/B) \subseteq \mathrm{Ev}(a/C)$.

Remark 13.7 *Let $\varphi(x, y) \in L$ and assume* $\mathrm{alt}(\varphi)$ *is finite. For any tuple b, for any set A, for any Ehrenfeucht-Mostowski set Φ we may choose the least $k_{\varphi,b} < \omega$ such that in any A-indiscernible infinite sequence with Ehrenfeucht-Mostowski set Φ $\varphi(x, b)$ has at most $k_{\varphi,b}$ alternations. This number can always be realized in any order type of an infinite sequence: for any infinite linearly ordered set I there is an A-indiscernible sequence $a = (a_i : i \in I)$ with Ehrenfeucht-Mostowski set Φ such that $\varphi(x, b)$ has $k_{\varphi,b}$ alternations in a.*

Proof: By compactness. If $p(x) = \mathrm{tp}(b/A)$, $\psi(x) \in p$, and $i_1 < \ldots < i_{k_{\varphi,b}}$ it is enough to find some A-indiscernible sequence $(a_i : i \in I)$ with Ehrenfeucht-Mostowski set Φ and some $c \models \psi$ such that $\models \varphi(a_{i_j}, c) \leftrightarrow \neg\varphi(a_{i_{j+1}}, c)$ for all $j = 1, \ldots, k_{\varphi,b} - 1$ and this is clearly possible since the formula

$$\exists x(\psi(x) \wedge \bigwedge_{1 \leq j < k_{\varphi,b}} \varphi(x_j, x) \leftrightarrow \neg\varphi(x_{j+1}, x))$$

belongs to Φ. □

Lemma 13.8 *Assume T has* NIP *and let I be linearly ordered, without last element. Let $a = (a_i : i \in I)$ be A-special and let $B \supseteq A$. For any $\varphi(x, y) \in L$, for any $b \in B$, since* $\mathrm{alt}(\varphi) < \infty$ *we may choose the least $k_{\varphi,b} < \omega$ such that in any $a' \equiv_A a$, $\varphi(x, b)$ has at most $k_{\varphi,b}$ alternations. Then $\varphi(x, b) \in \mathrm{Ev}(a/B)$ if and only if there is some $a' \equiv_A a$ such that $\varphi(x, b)$ has $k_{\varphi,b}$ alternations in a' and $\varphi(x, b) \in \mathrm{Av}(a'/B)$.*

Proof: Assume $\varphi(x, b) \in \mathrm{Ev}(a/B)$. Choose $a' \equiv_A a$ with biggest possible $k_{\varphi,b}$. There is some ω-sequence c such that $a'^\frown c$ is A-indiscernible and $\varphi(x, b) \in \mathrm{Av}(a'^\frown c/B)$. By choice of a' (and Remark 13.7), $\varphi(x, b) \in \mathrm{Av}(a'/B)$. For the other direction, let $a' \equiv_A a$ such that $\varphi(x, b)$ has $k_{\varphi,b}$ alternations in a' and $\varphi(x, b) \in \mathrm{Av}(a'/B)$. Let $d \equiv_A a$. Since $a' \equiv_A d$ and a' is A-special, there is some ω-sequence c such that $a' \frown c$ and $d^\frown c$ are A-indiscernible. Then $\varphi(x, b) \in \mathrm{Av}(a'^\frown c/A) = \mathrm{Av}(c/A) = \mathrm{Av}(d^\frown c/A)$. □

Lemma 13.9 *Assume T has* NIP *and let I be linearly ordered, without last element. If $a = (a_i : i \in I)$ is A-special, $\varphi(x) \in \mathrm{Ev}(a/B)$ and $\psi(x) \in \mathrm{Ev}(a/B)$, then $(\varphi(x) \wedge \psi(x)) \in \mathrm{Ev}(a/B)$*

Proof: Choose $a' \equiv_A a$ with maximal alternation number for $\varphi(x)$, $a'' \equiv_A a$ with maximal alternation number for $\psi(x)$, and $a''' \equiv_A a$ with maximal alternation number for $(\varphi(x) \wedge \psi(x))$. By Proposition 13.4, there is some ω-sequence c such that $a'^\frown c$, $a''^\frown c$ and $a'''^\frown c$ are indiscernible over A. Since $\varphi(x), \psi(x) \in \mathrm{Av}(c/B)$, $(\varphi(x) \wedge \psi(x)) \in \mathrm{Av}(a'''^\frown c/B)$ and hence $(\varphi(x) \wedge \psi(x)) \in \mathrm{Ev}(a/B)$. □

Proposition 13.10 *Assume T has* NIP *and let I be linearly ordered, without last element. For any A-special $a = (a_i : i \in I)$, for any $B \supseteq A$, $\mathrm{Ev}(a/B) \in S(B)$.*

Proof: Let $\varphi(x, y) \in L$ and $b \in B$. By Lemma 13.8, $\varphi(x, b) \in \mathrm{Ev}(a/B)$ or $\neg\varphi(x, b) \in \mathrm{Ev}(a/B)$. Now assume $\varphi(x, b) \in \mathrm{Ev}(a/B)$ and $\neg\varphi(x, b) \in \mathrm{Ev}(a/B)$. Choose some $a' \equiv_A a$ with biggest possible alternation number $k_{\varphi,b}$. Without loss of generality, $\varphi(x, b)$ holds in a final segment of a'. Since $\neg\varphi(x, b) \in \mathrm{Ev}(a/B)$, there is some ω-sequence c such that $a'^\frown c$ is A-indiscernible and $\neg\varphi(x, b) \in \mathrm{Av}(a'^\frown c/A)$. By Remark 13.7, this contradicts the choice of $k_{\varphi,b}$. □

Proposition 13.11 *Assume T has* NIP *and let I be linearly ordered, without last element. Let $a = (a_i : i \in I)$ be A-special. Then $\mathrm{Ev}(a/\mathfrak{C})$ does not split over A and $a_i \models \mathrm{Ev}(a/Aa_{<i})$ for all $i \in I$.*

Proof: Let $\varphi(x, y) \in L(A)$ and let $b \equiv_A b'$ be tuples such that $\varphi(x, b) \in \mathrm{Ev}(a/\mathfrak{C})$. Choose $c \equiv_A a$ with a maximal number of alternations of $\varphi(x, b)$ and choose c' such that $bc \equiv_A b'c'$. Then $c' \equiv_A a$ and it has a maximal number of alternations of $\varphi(x, b')$. By Lemma 13.8, $\varphi(x, b) \in \mathrm{Av}(c/Ab)$. Hence $\varphi(x, b') \in \mathrm{Av}(c'/Ab')$ and by Lemma 13.8 $\varphi(x, b') \in \mathrm{Ev}(a/Ab')$.

Now assume $\varphi(x_1, \ldots, x_n, x) \in L(A)$, $i_1 < \ldots < i_n < i$ and $\varphi(a_{i_1}, \ldots, a_{i_n}, x) \in \mathrm{Ev}(a/Aa_{<i})$. By definition of eventual type, there is some $c = (c_j : j < \omega)$ such that $a^\frown c$ is A-indiscernible and $\varphi(a_{i_1}, \ldots, a_{i_n}, x) \in \mathrm{Av}(a^\frown c/Aa_{<i})$. Then $\models \varphi(a_{i_1}, \ldots, a_{i_n}, c_j)$ for some j and by indiscernibility $\models \varphi(a_{i_1}, \ldots, a_{i_n}, a_i)$. This shows that $a_i \models \mathrm{Ev}(a/Aa_{<i})$. □

Corollary 13.12 *Assume T has* NIP *and let I be linearly ordered, without last element. A sequence $a = (a_i : i \in I)$ is A-special if and only if there is a global type $\mathfrak{p}$ that does not split over A and $a_i \models \mathfrak{p} \restriction Aa_{<i}$ for all $i \in I$. The global type $\mathfrak{p}$ is* $\mathrm{Ev}(a/\mathfrak{C})$.

Proof: From left to right use Proposition 13.11 with $\mathfrak{p} = \mathrm{Ev}(a/\mathfrak{C})$. For the other direction, it is straightforward that a is A-indiscernible. Assume $a' \equiv_A a$ and let $c \models \mathfrak{p} \restriction Aaa'$. Then $\mathfrak{p} \restriction a = \mathrm{Av}(a/Aa)$ and $\mathfrak{p} \restriction a' = \mathrm{Av}(a'/Aa')$, and hence $a^\frown(c)$ and $a'^\frown(c)$ are A-indiscernible. □

Corollary 13.13 (Strong Borel Definability) *Assume T has* NIP *and the global type $\mathfrak{p}$ does not split over A. For each $\varphi(x, y) \in L$, $\{b : \varphi(x, b) \in \mathfrak{p}\}$ is a finite boolean combination of A-type-definable subsets.*

Proof: Let $a = (a_i : i < \omega)$ be a Morley sequence over A with global type $\mathfrak{p}$. By Corollary 13.12 $\mathfrak{p} = \mathrm{Ev}(a/\mathfrak{C})$. Let n_φ be the alternation number of $\varphi(x, y) \in L$. By Lemma 13.8, $\varphi(x, b) \in \mathrm{Ev}(a/\mathfrak{C})$ if and only if for some $n \leq n_\varphi$ there are $(a'_1, \ldots, a'_n) \models \mathfrak{p}^{(n)} \restriction A$ such that $\models \bigwedge_{1 \leq i < n}(\varphi(a'_i, b) \leftrightarrow \neg\varphi(a'_{i+1}, b))$ and $\models \varphi(a'_n, b)$ and there are not $(a'_1, \ldots, a'_{n+1}) \models \mathfrak{p}^{(n+1)} \restriction A$ such that $\models \bigwedge_{1 \leq i < n+1}(\varphi(a'_i, b) \leftrightarrow \neg\varphi(a'_{i+1}, b))$. □

14 Weakly special sequences

Definition 14.1 An infinite sequence $a = (a_i : i \in I)$ is *weakly A-special* if it is A-indiscernible and every $b = (b_i : i \in I) \overset{\mathrm{Ls}}{\equiv}_A (a_i : i \in I)$ can be extended to an A-indiscernible sequence $b^\frown c$ by adding an ω-sequence c such that also $a^\frown c$ is A-indiscernible.

Lemma 14.2 *Let a, c be infinite sequences and assume $a^\frown c$ is A-indiscernible. If a is weakly A-special, then also $a^\frown c$ is weakly A-special.*

Proof: It is a modification of the proof of Lemma 13.2. Let $a = (a_i : i \in I)$ and $c = (c_j : j \in J)$, let $b = a^\frown c = (b_i : i \in I \cup J)$ and let $\Phi(x_i : i < \omega)$ be the Ehrenfeucht-Mostowski set over A of the A-indiscernible sequence $a = (a_i : i \in I)$. Assume $b \overset{\mathrm{Ls}}{\equiv}_A b' = (b'_j : j \in I \cup J)$. Let $\varphi(x_1, \ldots, x_n; y_1, \ldots, y_m) \in \Phi$ and $j_1 < \ldots < j_n$ in $I \cup J$. It is enough to check that $\varphi(b_{j_1}, \ldots, b_{j_n}, y_1, \ldots, y_m) \wedge \varphi(b'_{j_1}, \ldots, b'_{j_n}, y_1, \ldots, y_m)$ is consistent. Choose $i_1 < \ldots < i_n$ in I. Notice that $a_{i_1}, \ldots, a_{i_n} \overset{\mathrm{Ls}}{\equiv}_A b_{j_1}, \ldots, b_{j_n}$. Since a is weakly A-special, whenever $a_{i_1}, \ldots, a_{i_n} \overset{\mathrm{Ls}}{\equiv}_A d_1, \ldots, d_n$ then

$$\models \exists y_1, \ldots, y_m(\varphi(a_{i_1}, \ldots, a_{i_n}, y_1, \ldots, y_m) \wedge \varphi(d_1, \ldots, d_n, y_1, \ldots, y_m)).$$

By using some $f \in \mathrm{Autf}(\mathfrak{C}/A)$ sending $a_{i_1}, \ldots, a_{i_n}$ to $b_{j_1}, \ldots, b_{j_n}$, we see that whenever $b_{j_1}, \ldots, b_{j_n} \overset{\mathrm{Ls}}{\equiv}_A d_1, \ldots, d_n$ then also

$$\models \exists y_1, \ldots, y_m(\varphi(b_{j_1}, \ldots, b_{j_n}, y_1, \ldots, y_m) \wedge \varphi(d_1, \ldots, d_n, y_1, \ldots, y_m)).$$

In particular, $\models \exists y_1, \ldots, y_m(\varphi(b_{j_1}, \ldots, b_{j_n}, y_1, \ldots, y_m) \wedge \varphi(b'_{j_1}, \ldots, b'_{j_n}, y_1, \ldots, y_m))$ □

Fact 14.3 *(Reminding) KP-equivalence is finitary in the following sense: if* $a = (a_i : i \in I)$*, and* $b = (b_i : i \in I)$*, then* $a \overset{\text{KP}}{\equiv}_A b$ *if and only if* $a \restriction I_0 \overset{\text{KP}}{\equiv}_A b \restriction I_0$ *for all finite* $I_0 \subseteq I$*.*

Definition 14.4 *Assume* I *is an infinite linearly ordered set. Let* $a = (a_i : i \in I)$ *be weakly* A*-special. The* eventual type of a over $B \supseteq A$, $\mathrm{Ev}_A(a/B)$*, is the set of formulas* $\varphi(x) \in L(B)$ *such that for any* $b \overset{\text{Ls}}{\equiv}_A a$ *there is some* ω*-sequence* c *such that* $b^\frown c$ *is* A*-indiscernible and* $\varphi(x) \in \mathrm{Av}(b^\frown c/B)$*. We will see that this definition coincides with the older one if* a *is* A*-special. As in the previous case, usually we omit* A *and write* $\mathrm{Ev}(a/B)$*.*

Remark 14.5 *Let* $\varphi(x, y) \in L$ *and assume* $\mathrm{alt}(\varphi)$ *is finite. Let* b *be an* n*-tuple, let any set* A*, and assume* $a = (a_i : i \in I)$ *is* A*-indiscernible. We may choose the maximal* $k_{\varphi,b} < \omega$ *such that for any sequence* c *such that* $a^\frown c$ *is* A*-indiscernible* $\varphi(x, b)$ *has* $k_{\varphi,b}$ *alternations. This number can always be realized in a sequence* $a' \overset{\text{Ls}}{\equiv}_A a$*: there is some* $a' \overset{\text{Ls}}{\equiv}_A a$ *such that* $\varphi(x, b)$ *has* $k_{\varphi,b}$ *alternations in* a'*.*

Proof: Choose a model $M \supseteq A$. Let $p(x) = \mathrm{tp}(b/M)$ and let $q(x_i : i \in I) = \mathrm{tp}(a/M)$. If $\psi(x) \in p$, and $i_1 < \ldots < i_{k_{\varphi,b}} \in I$ then

$$\{\psi(x) \wedge \bigwedge_{1 \leq j < k_{\varphi,b}} \varphi(x_{i_j}, x) \leftrightarrow \neg\varphi(x_{i_{j+1}}, x)\} \cup q(x_i : i \in I)$$

is consistent. Hence for some $b' \equiv_M b$, for some $a' \equiv_M a$, $\models \bigwedge_{1 \leq j < k_{\varphi,b}} \varphi(a'_{i_1}, b') \leftrightarrow \neg\varphi(a_{i_{j+1}}, b')$. Let a'' be such that $b'a' \equiv_M ba''$. Then $a \overset{\text{Ls}}{\equiv}_A a''$ and $\varphi(x, b)$ has $k_{\varphi,b}$ alternations in a''. □

Lemma 14.6 *Assume* T *has* NIP*, and let* I *be linearly ordered, without last element. Let* $a = (a_i : i \in I)$ *be weakly* A*-special and let* $B \supseteq A$*. For any* $\varphi(x, y) \in L$*, for any* $b \in B$*, since* $\mathrm{alt}(\varphi) < \infty$ *we may choose the maximal* $k_{\varphi,b} < \omega$ *such that in some* $a' \overset{\text{Ls}}{\equiv}_A a$*,* $\varphi(x, b)$ *has* $k_{\varphi,b}$ *alternations. Then* $\varphi(x, b) \in \mathrm{Ev}(a/B)$ *if and only if there is some* $a' \overset{\text{Ls}}{\equiv}_A a$ *such that* $\varphi(x, b)$ *has* $k_{\varphi,b}$ *alternations in* a' *and* $\varphi(x, b) \in \mathrm{Av}(a'/B)$*.*

Proof: Assume $\varphi(x, b) \in \mathrm{Ev}(a/B)$. Choose $a' \overset{\text{Ls}}{\equiv}_A a$ with biggest possible $k_{\varphi,b}$. There is some ω-sequence c such that $a'^\frown c$ is A-indiscernible and $\varphi(x, b) \in \mathrm{Av}(a'^\frown c/B)$. By choice of a' (and Remark 14.5), $\varphi(x, b) \in \mathrm{Av}(a'/B)$. For the other direction, let $a' \overset{\text{Ls}}{\equiv}_A a$ such that $\varphi(x, b)$ has $k_{\varphi,b}$ alternations in a' and $\varphi(x, b) \in \mathrm{Av}(a'/B)$. Let $d \overset{\text{Ls}}{\equiv}_A a$. Since $a' \overset{\text{Ls}}{\equiv}_A d$ and a' is weakly A-special, there is some ω-sequence c such that $a' \frown c$ and $d^\frown c$ are A-indiscernible. Then $\varphi(x, b) \in \mathrm{Av}(a'^\frown c/A) = \mathrm{Av}(c/A) = \mathrm{Av}(d^\frown c/A)$. □

Proposition 14.7 *Assume* T *has* NIP*. Let* I *be linearly ordered, without last element. For any weakly* A*-special* $a = (a_i : i \in I)$*, for any* $B \supseteq A$*,* $\mathrm{Ev}(a/B) \in S(B)$*.*

Proof: Like Proposition 13.10 but using now Lemma 14.6 and Remark 14.5. □

Remark 14.8 *Assume* T *has* NIP*. If* a *is* A*-special, then its eventual type* $\mathrm{Ev}_s(a/B)$ *as special sequence and its eventual type* $\mathrm{Ev}_{ws}(a/B)$ *as weakly special sequence coincide.*

Proof: By Proposition 14.7, since clearly $\mathrm{Ev}_s(a/B) \subseteq \mathrm{Ev}_{ws}(a/B)$. □

Proposition 14.9 *Assume* T *has* NIP*, and let* I *be linearly ordered without last element. Let* $a = (a_i : i \in I)$ *be weakly* A*-special. Then* $\mathrm{Ev}(a/\mathfrak{C})$ *does not Lascar split over* A *and* $a_i \models \mathrm{Ev}(a/Aa_{<i})$ *for all* $i \in I$*.*

Proof: Like the proof of Proposition 13.11 but using Lascar strong types. □

Lemma 14.10 *Assume T has* NIP*, and let I be linearly ordered. If $a = (a_i : i \in I)$ is a Morley sequence over A and $b = (b_i : i \in I) \overset{\text{Ls}}{\equiv}_A (a_i : i \in I)$, then there is some tuple c such that $a^\frown(c)$ and $b^\frown(c)$ are A-indiscernible.*

Proof: We may assume I does not have a last element. Let $f \in \mathrm{Aut}(\mathfrak{C}/A)$ be such that $f(a) = b$. Let $\mathfrak{p}$ be the global type associated to a as in Proposition 12.4 and let $\mathfrak{q}$ be the corresponding type for b. Then $\mathfrak{p}^f = \mathfrak{q}$. Since $\mathfrak{p}$ does not fork over a, it does not Lascar-split over A and therefore $\mathfrak{p}^f = \mathfrak{p}$. Hence $\mathfrak{p} = \mathfrak{q}$. Let $c \models \mathfrak{p} \restriction Aab$. Since $\mathfrak{p} \restriction Aa = \mathrm{Av}(a/Aa)$ and $\mathfrak{p} \restriction Ab = \mathrm{Av}(b/Ab)$, by By Remark 6.5 $a^\frown(c)$ and $b^\frown(c)$ are A-indiscernible. □

Lemma 14.11 *Assume T has* NIP*, and let I be linearly ordered. Assume $a = (a_i : i \in I)$ is an infinite Morley sequence over A. If $b = (b_i : i \in I) \overset{\text{Ls}}{\equiv}_A (a_i : i \in I)$ and c is a tuple such that $a^\frown(c)$ and $b^\frown(c)$ are A-indiscernible, then $a^\frown(c) \overset{\text{Ls}}{\equiv}_A b^\frown(c)$.*

Proof: Since $a^\frown(c)$ and $b^\frown(c)$ are again Morley sequences over A, by Proposition 12.5, $a^\frown(c) \mathop{\smile\!\!\!\!\!\mid}_A A$ and $b^\frown(c) \mathop{\smile\!\!\!\!\!\mid}_A A$, and by Corollary 11.9 it is enough to show that $a^\frown(c) \overset{\text{KP}}{\equiv}_A b^\frown(c)$. As stated in Fact 14.3, it suffices to show that all finite subsequences have the same KP-type over A and this is clear since we can find corresponding finite tuples in a and b with same KP-type over A. □

Proposition 14.12 *Assume T has* NIP*, and let I be linearly ordered. If $a = (a_i : i \in I)$ is a Morley sequence over A, then a is weakly special over A.*

Proof: Let $b \overset{\text{Ls}}{\equiv}_A a$. By Lemma 14.10 there is some tuple c such that $a^\frown(c)$ and $b^\frown(c)$ are A-indiscernible. By Lemma 14.11, $a^\frown(c) \overset{\text{Ls}}{\equiv}_A b^\frown(c)$. Since $a^\frown(c)$ and $b^\frown(c)$ are Morley sequences over A, the process can be iterated and we can obtain an ω-sequence c such that $a^\frown c$ and $b^\frown c$ are A-indiscernible. □

Corollary 14.13 *Assume T has* NIP. *If $a = (a_i : i \in I)$ is a Morley sequence over A, then the global type of a is* $\mathrm{Ev}(a/\mathfrak{C})$.

Proposition 14.14 *If $a = (a_i : i < \omega)$ is a Morley sequence over A and $B \supseteq A$, there is a Morley sequence $b = (b_i : i < \omega)$ over B such that $a \equiv_A b$.*

Proof: Let α be the length of each a_i, let $\kappa = |B| + |T| + |\alpha|$ and $\lambda = \beth_{(2^\kappa)^+}$. Extend a to an A-indiscernible sequence $(a_i : i < \lambda)$. It is also a Morley sequence over A. Construct inductively a sequence $(a'_i : i < \lambda)$ such that for all $i < \lambda$, $a_{<i} \equiv_A a'_{<i}$ and $a'_i \mathop{\smile\!\!\!\!\!\mid}_A Ba'_{<i}$. To obtain a'_i we choose some $f \in \mathrm{Aut}(\mathfrak{C}/A)$ such that $f(a_{<i}) = a'_{<i}$. Since $p(x) = \mathrm{tp}(a_i/Aa_{<i})$ does not fork over A, its conjugate $p^f(x) \in S(Aa'_{<i})$ does not fork over A and hence it has an extension $q(x) \in S(Ba'_{<i})$ which does not fork over A. We take as a'_i a realization of q. Then $a_{<i}a_i \equiv_A a'_{<i}a'_i$ and $a'_i \mathop{\smile\!\!\!\!\!\mid}_A Ba'_{<i}$. There is a B-indiscernible sequence $b = (b_i : i < \omega)$ such that for each $n < \omega$ there are $i_0 < \ldots < i_n < \lambda$ such that $b_0, \ldots, b_n \equiv_B a'_{i_0}, \ldots, a'_{i_n}$. Then

$$b_0, \ldots, b_n \equiv_A a'_{i_0}, \ldots, a'_{i_n} \equiv_A a_{i_0}, \ldots, a_{i_n} \equiv_A a_0, \ldots, a_n$$

and therefore $a \equiv_A b$. Since $a'_{i_n} \mathop{\smile\!\!\!\!\!\mid}_A Ba'_{<i_n}$, we see that $b_n \mathop{\smile\!\!\!\!\!\mid}_A Bb_{<n}$ and thus $b_n \mathop{\smile\!\!\!\!\!\mid}_B b_{<n}$. This shows that b is a Morley sequence over B. □

Proposition 14.15 *If $a = (a_i : i < \omega)$ is a Morley sequence over A and $B \supseteq A$, there is a Morley sequence $b = (b_i : i < \omega)$ over B such that $a \overset{\mathrm{KP}}{\equiv}_A b$. Assuming* NIP *we can obtain $a \overset{\mathrm{Ls}}{\equiv}_A b$.*

Proof: It is an elaboration of the proof of Proposition 14.14, so we only point out the modifications. We extend $(a_i : i < \omega)$ to the A-indiscernible sequence $(a_i : i < \lambda)$, we choose a model $M \supseteq B$, and we construct inductively the sequence $(a'_i : i < \lambda)$ in such a way that $a_{<i} \overset{\mathrm{KP}}{\equiv}_A a'_{<i}$ and $a'_i \mathop{\smile\!\!\!\!\!\!\mid}_A Ma'_{<i}$. We explain how to obtain a'_i. Since $a_i \mathop{\smile\!\!\!\!\!\!\mid}_A a_{<i}$, $\mathrm{tp}(a_i/Aa_{<i})$ has a extension over $\mathrm{bdd}(Aa_{<i})$ that does not fork over A. Since all extensions of $\mathrm{tp}(a_i/Aa_{<i})$ over $\mathrm{bdd}(Aa_{<i})$ are $Aa_{<i}$-conjugate, no such extension forks over A. Hence $a_i \mathop{\smile\!\!\!\!\!\!\mid}_A \mathrm{bdd}(Aa_{<i})$ and, in particular, $a_i \mathop{\smile\!\!\!\!\!\!\mid}_A \mathrm{bdd}(A)a_{<i}$. By the induction hypothesis, $a_{<i} \overset{\mathrm{KP}}{\equiv}_A a'_{<i}$ and hence $a_{<i} \equiv_{\mathrm{bdd}(A)} a'_{<i}$. Choose an automorphism $f \in \mathrm{Aut}(\mathfrak{C}/\mathrm{bdd}(A))$ sending $a_{<i}$ to $a'_{<i}$. If $p(x) = \mathrm{tp}(a_i/\mathrm{bdd}(A)a_{<i})$, p^f has an extension over $Ma'_{<i}$ that does not fork over A. We take as a'_i a realization of this extension.

Finally we obtain the M-indiscernible sequence $b = (b_i : i < \omega)$. It is a Morley sequence over B. For every $n < \omega$, there are $i_0 < \ldots < i_n < \lambda$ such that $b_0, \ldots, b_n \equiv_M a'_{i_0}, \ldots, a'_{i_n}$. Therefore

$$b_0, \ldots, b_n \overset{\mathrm{Ls}}{\equiv}_A a'_{i_0}, \ldots, a'_{i_n} \overset{\mathrm{KP}}{\equiv}_A a_{i_0}, \ldots, a_{i_n} \overset{\mathrm{Ls}}{\equiv}_A a_0, \ldots, a_n.$$

Hence $a \overset{\mathrm{KP}}{\equiv}_A b$. If T is NIP, by Proposition 12.5, $a \mathop{\smile\!\!\!\!\!\!\mid}_A A$, and Corollary 11.9 gives $a \overset{\mathrm{Ls}}{\equiv}_A b$. □

Theorem 14.16 *Assume T has* NIP. *The following are equivalent for $a = (a_i : i < \omega)$.*

1. *a is weakly special over A.*
2. *a is A-indiscernible and there is a global type $\mathfrak{p}$ that does not fork over A and $a_i \models \mathfrak{p} \upharpoonright Aa_{<i}$ for all $i < \omega$.*
3. *a is a Morley sequence over A.*
4. *For some Lascar-complete set B over A there is a type $p(x) \in S(B)$ that does not fork over A and $b \models p^{(\omega)_A}$, for some $b \equiv_A a$.*
5. *For some Lascar-complete set B over A there is a type $p(x) \in S(B)$ that does not fork over A and $b \models p^{(\omega)_A}$, for some $b \overset{\mathrm{Ls}}{\equiv}_A a$.*

Proof: *1* ⇒ *2*. By Proposition 14.9.

2 ⇔ *3*. By Proposition 12.4.

3 ⇒ *5*. By Proposition 14.15.

5 ⇒ *4*. Clear.

4 ⇒ *3*. By Lemma 11.6.

3 ⇒ *1*. By Proposition 14.12. □

Corollary 14.17 *Assume T has* NIP. *If $a = (a_i : i \in I)$ is a Morley sequence over A, then for any family $(b^i : i < \lambda)$ where $b^i \overset{\mathrm{Ls}}{\equiv}_A a$, for any linearly ordered set J there is some sequence $c = (c_j : j \in J)$ such that every $b^i {}^\frown c$ is a Morley sequence over A.*

Proof: By Lemma 14.11 and compactness. □

Example 14.18 1. A Morley sequence which is not special (in an ω-stable theory). Let T be the theory of an equivalence relation E with exactly two classes, both infinite. Choose $(a_i : i < \omega)$, different elements in one E-class. It is Morley (over $\emptyset$) but not special.

2. Indiscernible sequences that are not Morley (again in an ω-stable theory). Let T be the theory of an equivalence relation E with infinitely many classes, all infinite. Take $(a_i : i < \omega)$, a sequence of different elements in an E-class. It is indiscernible (over $\emptyset$) but not it is not Morley. If we take $(b_i : i < \omega)$ where each b_i is in a different E-class, then it is special over $\emptyset$ (and hence Morley).

3. Eventual types and average types. Let T be the theory of the dense linear order without endpoints. Let $a = (a_i : i < \omega)$ be a strictly increasing sequence and let (X, Y) be the cut defined by a in the monster model. The sequence is special over $\emptyset$. Then $\mathrm{Av}(a/\mathfrak{C})$ is the type of the cut (X, Y) while $\mathrm{Ev}(a/\mathfrak{C})$ is the type $+\infty$ (the type of an element greater than every element of $\mathfrak{C}$). Notice that if we choose $b \in Y$, then a is b-special and $\mathrm{Ev}_b(a/\mathfrak{C})$ is now the type b^- of the left part of the cut determined by b.

15 Generically stable types

Proposition 15.1 *Assume T has* NIP *and $\mathfrak{p}$ does not fork over A.*

1. *If $\mathfrak{p}$ is definable, then it is definable over* $\mathrm{acl}^{\mathrm{eq}}(A)$.
2. *If $\mathfrak{p}$ is finitely satisfiable in some $M \supseteq A$, then it is finitely satisfiable in every $M \supseteq A$.*

Proof: 1. By Proposition 11.13, $\mathfrak{p}$ does not KP-split over A, that is, $\mathfrak{p}$ is $\mathrm{bdd}(A)$-invariant. Let $\varphi(x, y) \in L$ and let $c \in \mathfrak{C}^{\mathrm{eq}}$ be the canonical parameter of a definition of $\{a : \varphi(x, a) \in \mathfrak{p}\}$. Since $c \in \mathrm{bdd}(A)$ and c is an imaginary, $c \in \mathrm{acl}^{\mathrm{eq}}(A)$.

2. Fix $N \supseteq A$ such that $\mathfrak{p}$ is finitely satisfiable in N and let $M \supseteq A$. Let $\varphi(x, y) \in L$, and assume $\varphi(x, a) \in \mathfrak{p}$. Choose $N' \equiv_M N$ such that $\mathrm{tp}(N'/Ma)$ coheirs from M. Since $\mathfrak{p}$ is M-invariant, $\mathfrak{p}$ is finitely satisfiable in N'. Then there is some $b \in N'$ such that $\models \varphi(b, a)$. It follows that for some $b' \in M$, $\models \varphi(b', a)$. □

Proposition 15.2 *If $\mathfrak{p}$ is finitely satisfiable in M and it is definable over M and the sequence $a = (a_i : i < \omega)$ is M-indiscernible and satisfies $a_i \models \mathfrak{p} \restriction Ma_{<i}$ for all $i < \omega$, then a is totally indiscernible over M.*

Proof: It is enough to show that for every $n < \omega$ every permutation of $\{a_0, \ldots, a_n\}$ is elementary over M. Since a permutation is a product of transpositions of consecutive elements, it is enough to prove that for all $i < n$,

$$a_{<i}a_ia_{i+1}a_{i+2}, \ldots, a_n \equiv_M a_{<i}a_{i+1}a_ia_{i+2}, \ldots, a_n. \tag{1}$$

For this we will first prove that for all $i \leq n$,

$$a_i \models \mathfrak{p} \restriction Ma_{<i}a_{i+1} \ldots a_n \tag{2}$$

Let us check that (2) implies (1). Assume (2). Notice that $a_{n+1} \models \mathfrak{p} \restriction Ma_{\leq i}a_{i+2} \ldots a_n$ and hence

$$a_{<i}a_ia_{i+1}a_{i+2}, \ldots, a_n \equiv_M a_{<i}a_ia_{n+1}a_{i+2}, \ldots, a_n.$$

By indiscernibility over M:

$$a_{<i}a_ia_{n+1}a_{i+2},\dots,a_n \equiv_M a_{<i}a_{i+1}a_{n+1}a_{i+2},\dots,a_n.$$

Again by (2) and because $a_{n+1} \models \mathfrak{p} \upharpoonright Ma_{<i}a_{i+1}\dots a_n$,

$$a_{<i}a_{i+1}a_{n+1}a_{i+2},\dots,a_n \equiv_M a_{<i}a_{i+1}a_ia_{i+2},\dots,a_n.$$

Now we prove (2). Since $\mathfrak{p}$ is M-definable, $\mathfrak{p} \upharpoonright Ma_{<i}a_{i+1}\dots a_n$ is the unique M-definable extension of $\mathfrak{p} \upharpoonright M$ over $Ma_{<i}a_{i+1}\dots a_n$ and it is therefore the unique heir of $\mathfrak{p} \upharpoonright M$ over $Ma_{<i}a_{i+1}\dots a_n$. We must check that $\mathrm{tp}(a_i/Ma_{<i}a_{i+1}\dots a_n)$ is a heir of $\mathrm{tp}(a_i/M)$ or, in other terms, that $\mathrm{tp}(a_{<i}a_{i+1},\dots,a_n/Ma_i)$ coheirs from M. We start checking that

$$\mathrm{tp}(a_{i+1},\dots,a_n/Ma_{\leq i}) \text{ coheirs from } M. \tag{3}$$

Let $\varphi(x_{i+1},\dots,x_n) \in L(Ma_{\leq i})$ be such that $\models \varphi(a_{i+1},\dots,a_n)$. Since $\mathfrak{p}$ is finitely satisfiable in M and $a_n \models \mathfrak{p}Ma_{<n}$, there is some $a'_n \in M$ such that $\models \varphi(a_{i+1},\dots,a_{n-1},a'_n)$. By iteration we obtain $a'_{i+1},\dots,a'_n \in M$ such that $\models \varphi(a'_{i+1},\dots,a'_n)$.

Now we finish the proof checking that

$$\mathrm{tp}(a_{<i}a_{i+1},\dots,a_n/Ma_i) \text{ coheirs from } M. \tag{4}$$

Let $\varphi(x_{<i},x_{i+1},\dots,x_n,x_i) \in L(M)$ be such that $\models \varphi(a_{<i},a_{i+1},\dots,a_n,a_i)$. By (3) there are $a'_{i+1},\dots,a'_n \in M$ such that $\models \varphi(a_{<i},a'_{i+1},\dots,a'_n,a_i)$. Since $\mathrm{tp}(a_i/Ma_{<i} = \mathfrak{p} \upharpoonright Ma_{<i}$ is definable over M, there is some $\theta(x_{<i},x_{i+1},\dots,x_n) \in L(M)$ such that for all $b_{<i},b_{i+1},\dots,b_n$ in $Ma_{<i}$

$$\models \theta(b_{<i},b_{i+1},\dots,b_n) \text{ if and only if } \varphi(b_{<i},b_{i+1},\dots,b_n,x) \in \mathrm{tp}(a_i/Ma_{<i}). \tag{5}$$

In particular

$$\models \theta(a_{<i},a'_{i+1},\dots,a'_n) \text{ if and only if } \varphi(a_{<i},a'_{i+1},\dots,a'_n,x) \in \mathrm{tp}(a_i/Ma_{<i})$$

and therefore $\models \theta(a_{<i},a'_{i+1},\dots,a'_n)$. It follows that

$$M \models \exists x_{<i}\theta(x_{<i},a'_{i+1},\dots,a'_n)$$

and then there is some $a'_{<i} \in M$ such that $\models \theta(a'_{<i},a'_{i+1},\dots,a'_n)$ and by (5)

$$\varphi(a'_{<i},a'_{i+1},\dots,a'_n,x) \in \mathrm{tp}(a_i/Ma_{<i}),$$

that is, $\models \varphi(a'_{<i},a'_{i+1},\dots,a'_n,a_i)$. □

Lemma 15.3 *Let $a = (a_i : i < \omega)$ and $b = (b_i : i < \omega)$. If $a^\frown b$ is A-indiscernible and a is totally indiscernible over A, then $a^\frown b$ is totally indiscernible over A. Moreover, if T has* NIP, *then* $\mathrm{Av}(a^\frown b/\mathfrak{C}) = \mathrm{Av}(a/\mathfrak{C})$.

Proof: Let c, c' be finite subsequences of $a^\frown b$. Assume they have the same length and they do not contain repetitions. Find a subsequence d of a with the same order type as c and a suptuple d' of d with the same order type as c'. Since $a^\frown b$ is A-indiscernible, $c \equiv_A d$ and $c' \equiv_A d'$. Since a is totally indiscernible over A, $d \equiv_A d'$. Hence $c \equiv_A c'$.

Assume now T has NIP. By Proposition 6.2, if $c = (c_i : i < \omega)$ is totally indiscernible, then $\varphi(x) \in \mathrm{Av}(c/\mathfrak{C})$ if and only if $\{i < \omega : \models \varphi(c_i)\}$ is infinite (equivalently, cofinite). Hence if $\varphi(x) \in \mathrm{Av}(a/\mathfrak{C})$ then also $\varphi(x) \in \mathrm{Av}(a^\frown b/\mathfrak{C})$. This shows $\mathrm{Av}(a/\mathfrak{C}) \subseteq \mathrm{Av}(a^\frown b/\mathfrak{C})$ and therefore $\mathrm{Av}(a/\mathfrak{C}) = \mathrm{Av}(a^\frown b/\mathfrak{C})$. □

Proposition 15.4 *Assume T has* NIP*. Let $a = (a_i : i < \omega)$ be a Morley sequence over A. If a is totally indiscernible, then $\mathrm{Av}(a/\mathfrak{C})$ does not fork over A and it is the global type associated to a.*

Proof: Let $\mathfrak{p}$ be the global type associated to a, that is, $\mathfrak{p}$ is the unique global type that does not fork over A and satisfies $a_i \models \mathfrak{p} \upharpoonright Aa_{<i}$ for all $i < \omega$. Since $a_i \models \mathrm{Av}(a/Aa_{<i})$, we only need to show that $\mathrm{Av}(a/\mathfrak{C})$ does not fork over A. Let $f \in \mathrm{Autf}(\mathfrak{C}/A)$ and let $f(a) = a' = (a'_i : i < \omega)$. It suffices to show that $\mathrm{Av}(a/\mathfrak{C}) = \mathrm{Av}(a'/\mathfrak{C})$. Since $a \overset{\mathrm{Ls}}{\equiv}_A a'$ and a is weakly special over A, there is an ω sequence $b = (b_i : i < \omega)$ such that $a^\frown b$ and $a'^\frown b$ are A-indiscernible. It is clear that $\mathrm{Av}(a^\frown b/\mathfrak{C}) = \mathrm{Av}(b/\mathfrak{C}) = \mathrm{Av}(a'^\frown b/\mathfrak{C})$. By Lemma 15.3, $\mathrm{Av}(a/\mathfrak{C}) = \mathrm{Av}(a^\frown b/\mathfrak{C})$ and $\mathrm{Av}(a'/\mathfrak{C}) = \mathrm{Av}(a'^\frown b/\mathfrak{C})$. □

Lemma 15.5 *Assume T has* NIP*, and assume $a = (a_i : i < \omega)$ is totally indiscernible over A. Then $\mathrm{Av}(a/\mathfrak{C})$ is definable over a: if $\varphi(x, y) \in L$, there is a number $n_\varphi < \omega$ such that for all c,*

$$\varphi(x, c) \in \mathrm{Av}(a/\mathfrak{C}) \text{ if and only if } \models \bigvee_{w \subseteq 2 \cdot n_\varphi,\ |w| = n_\varphi} \bigwedge_{i \in w} \varphi(a_i, c)$$

Proof: The number n_φ is given by Remark 6.3. In fact $n_\varphi = \mathrm{alt}(\varphi) + 2$. □

Proposition 15.6 *Assume T has* NIP*. Let $a = (a_i : i < \omega)$ be a Morley sequence over A and let $\mathfrak{p}$ be its associated global type. If for each $\varphi(x, y) \in L$ there is a number $n_\varphi < \omega$ such that $\mathfrak{p}$ is definable over a by*

$$\varphi(x, c) \in \mathfrak{p} \text{ if and only if } \models \bigvee_{w \subseteq 2 \cdot n_\varphi,\ |w| = n_\varphi} \bigwedge_{i \in w} \varphi(a_i, c)$$

then for every model $M \supseteq A$, $\mathfrak{p}$ is definable over M and it is finitely satisfiable in M.

Proof: Since $\mathfrak{p}$ is finitely satisfiable in a, it is finitely satisfiable in some model $M \supseteq A$. By Proposition 15.1, it is finitely satisfiable in every model $M \supseteq A$. Since $\mathfrak{p}$ is definable, by Proposition 15.1 it is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$. Hence it is definable over every $M \supseteq A$. □

Definition 15.7 A global type $\mathfrak{p}$ is *generically stable over A* if for some model $M \supseteq A$, $\mathfrak{p}$ is definable over M and it is finitely satisfiable in M.

Theorem 15.8 *If T has* NIP *and $\mathfrak{p}$ does not fork over A, the following are equivalent:*

1. *$\mathfrak{p}$ is generically stable over A.*
2. *For every model $M \supseteq A$, $\mathfrak{p}$ is definable over M and finitely satisfiable in M.*
3. *For every model $M \supseteq A$, every Morley sequence $(a_i : i < \omega)$ over M with associated global type $\mathfrak{p}$ is totally indiscernible over M.*
4. *Every (some) realization of $\mathfrak{p}^{(\omega)} \upharpoonright A$ is totally indiscernible over A.*
5. *For every $\varphi(x, y) \in L$ there is some number $n_\varphi < \omega$ such that for every (some) Morley sequence $(a_i : i < \omega)$ over A with global type $\mathfrak{p}$, $\mathfrak{p}$ is definable over a by*

$$\varphi(x, c) \in \mathfrak{p} \text{ if and only if } \models \bigvee_{w \subseteq 2 \cdot n_\varphi,\ |w| = n_\varphi} \bigwedge_{i \in w} \varphi(a_i, c)$$

Proof: *1* ⇔ *2*. By Proposition 15.1.

2 ⇒ *3*. By Proposition 15.2.

3 ⇒ *4*. Choose $M \supseteq A$ Lascar-complete over A and let $p = \mathfrak{p} \upharpoonright M$. Then p does not fork over A and $p^{(\omega)_A} = \mathfrak{p}^{(\omega)} \upharpoonright M$. If $a \models \mathfrak{p}^{(\omega)} \upharpoonright A$, then $a \equiv_A b$ for some $b \models p^{(\omega)_A}$, a Morley sequence over M. The associated global type of b is $\mathfrak{p}$. By *3* b is totally indiscernible over M. Hence a is totally indiscernible over A.

4 ⇒ *5*. By Proposition 15.4 and Lemma 15.5.

5 ⇒ *1*. By Proposition 15.6. □

Proposition 15.9 *Assume T has* NIP*. If $\mathfrak{p}$ is A-invariant and generically stable over A, then $\mathfrak{p} \upharpoonright A$ is stationary.*

Proof: Let $\mathfrak{q}$ be a nonforking extension of $\mathfrak{p} \upharpoonright A$. We will show that $\mathfrak{p}$ and $\mathfrak{q}$ have a common Morley sequence over A and from this, by Lemma 12.3, it will follow that $\mathfrak{p} = \mathfrak{q}$. Let $a = (a_i : i < \omega) \models \mathfrak{p}^{(\omega)} \upharpoonright A$ and let $b \models \mathfrak{q} \upharpoonright Aa$. Then a is a Morley sequence over A with global type $\mathfrak{p}$. By Theorem 15.8 and Proposition 15.4, $\mathfrak{p} = \mathrm{Av}(a/\mathfrak{C})$. We claim that for all $i < \omega$,

$$b \equiv_{Aa_{<i}} a_i.$$

We prove it by induction on i. It is clear for $i = 0$, since $\mathfrak{p} \upharpoonright A = \mathfrak{q} \upharpoonright A$. Let $\varphi(x_0, \ldots, x_{i+1}) \in L(A)$ and assume $\models \varphi(a_0, \ldots, a_i, b)$. Then $\varphi(a_0, \ldots, a_i, x) \in \mathfrak{q}$. If $j \geq i$, then $a_{<i}a_i \overset{\mathrm{Ls}}{\equiv}_A a_{<i}a_j$ and since $\mathfrak{q}$ does not Lascar-split over A, $\varphi(a_0, \ldots, a_{i-1}a_j, x) \in \mathfrak{q}$, that is $\models \varphi(a_0, \ldots, a_{i-1}a_j, b)$. Since $\mathfrak{p} = \mathrm{Av}(a/\mathfrak{C})$, $\varphi(a_0, \ldots, a_{i-1}, x, b) \in \mathfrak{p}$. By the induction hypothesis and A-invariance of $\mathfrak{p}$, $\varphi(a_0, \ldots, a_{i-1}, x, a_i) \in \mathfrak{p}$. Then $\models \varphi(a_0, \ldots, a_{i-1}, a_{i+1}, a_i)$. Since a is totally indiscernible over A, $\models \varphi(a_0, \ldots, a_{i-1}, a_i, a_{i+1})$. By the claim we get $\mathrm{tp}(a_i/Aa_{<i}) = \mathrm{tp}(b/Aa_{<i}) = \mathfrak{q} \upharpoonright Aa_{<i}$ and hence $\mathfrak{q}$ is the global type associated over A to the Morley sequence a. □

Theorem 15.10 *Assume T has* NIP *and $\mathfrak{p}$ is A-invariant. The following are equivalent:*

1. *$\mathfrak{p}$ is generically stable over A.*
2. *For every $B \supseteq A$, $\mathfrak{p} \upharpoonright B$ is stationary.*
3. *For every $n \geq 1$, for every $B \supseteq A$, $\mathfrak{p}^{(n)} \upharpoonright B$ is stationary.*

Proof: Note that if $\mathfrak{p}$ is generically stable over A, then it is generically stable over any $B \supseteq A$. Hence *1* ⇒ *2* follows from Proposition 15.9.

1 ⇒ *3*. Since $\mathfrak{p}$ is A-invariant, $\mathfrak{p}^{(n)}$ is A-invariant too. Notice that, by associativity of the product, $(\mathfrak{p}^{(n)})^{(m)} = \mathfrak{p}^{(n \cdot m)}$ and hence any realization of $(\mathfrak{p}^{(n)})^{(\omega)} \upharpoonright A$ is (after elimination of brackets) a realization of $\mathfrak{p}^{(\omega)} \upharpoonright A$. By point *4* of Theorem 15.8, $\mathfrak{p}^{(n)}$ is generically stable over A. By the previous paragraph, *3* follows from *1*.

2 ⇒ *1*. Let $a = (a_i : i < \omega)$ be a Morley sequence over A with global type $\mathfrak{p}$. By Proposition 12.4, $\mathfrak{p} \upharpoonright Aa = \mathrm{Av}(a/Aa)$. By Remark 6.4, $\mathrm{Av}(a/\mathfrak{C})$ does not fork over $B = Aa$. By stationarity, $\mathfrak{p} = \mathrm{Av}(a/\mathfrak{C})$. By Lemma 15.5 and point *5* of Theorem 15.8, $\mathfrak{p}$ is generically stable over A. □

Definition 15.11 If $p(x, y) \in S(A)$ we define $p^{-1}(y, x)$ as the type $\mathrm{tp}(ba/A)$ for any $ab \models p$. This is only well-defined when the separation of variables x, y is fixed. Note that p and p^{-1} share almost all model-theoretical properties. This notation extends the more familiar notation used for formulas: $\varphi^{-1}(y, x)$ is the formula $\varphi(x, y)$ with opposite separation of variables. Note that $p^{-1} = \{\varphi^{-1} : \varphi \in p\}$.

Lemma 15.12 [4] *Assume T has* NIP*. Let B be A-complete and assume $p(x) \in S(B)$ does not split over A. If $(p \otimes_A p)^{-1} = p \otimes_A p$, then every realization of $p^{(\omega)_A}$ is totally indiscernible over B.*

Proof: It is enough to prove that for every $n < \omega$, for every permutation π of $\{1, \ldots, n\}$, $(a_1, \ldots, a_n) \models p^{(n)_A}$ if and only if $(a_{\pi(1)}, \ldots, a_{\pi(n)}) \models p^{(n)_A}$. Since every such permutation is a composition of transpositions of consecutive elements and since the product of types is associative, it suffices to check that for all $n < \omega$, $(a_1, \ldots, a_n, a_{n+1}) \models p^{(n+1)_A}$ if and only if $(a_1, \ldots, a_{n-1}, a_{n+1}, a_n) \models p^{(n+1)_A}$. But this is clear since

$$(a_1, \ldots, a_n, a_{n+1}) \models p^{(n+1)_A} \Leftrightarrow a_{<n} \models p^{(n-1)_A} \text{ and } (a_n, a_{n+1}) \models (p \otimes_A p)|_A B a_{<n},$$

and

$$(a_n, a_{n+1}) \models (p \otimes_A p)|_A B a_{<n} \Leftrightarrow (a_{n+1}, a_n) \models ((p \otimes_A p)|_A B a_{<n})^{-1},$$

and

$$((p \otimes_A p)|_A B a_{<n})^{-1} = (p \otimes_A p)^{-1}|_A B a_{<n} = (p \otimes_A p)|B a_{<n}.$$

□

Theorem 15.13 *Assume T has* NIP*, $\mathfrak{p}$ is A-invariant and no type over A forks over A. Then $\mathfrak{p}$ is generically stable over A if and only if for all $n \geq 1$, $\mathfrak{p}^{(n)} \restriction A$ is stationary.*

Proof: One direction follows from Theorem 15.10. Assume then the right hand side and let us check that $\mathfrak{p}$ is generically stable over A. Choose $B \supseteq A$ A-complete and let $p = \mathfrak{p} \restriction A$. By Theorem 15.8 and Lemma 15.12 it suffices to prove that $(p \otimes_A p)^{-1} = p \otimes_A p$. We need some preparation. For $n < \omega$, let $p^{(-n)_A}$ be the $\mathrm{tp}(a_n, \ldots, a_1/A)$ for $(a_1, \ldots, a_n) \models p^{(n)_A}$ and let

$$p^{(\omega^*)_A} = \bigcup_{n<\omega} p^{-(n+1)_A}(x_0, \ldots, x_n).$$

Then $p^{(\omega^*)_A} \in S(A)$ and

$$(a_i : i < \omega) \models p^{(\omega^*)_A} \Leftrightarrow a_i \models p|_A B a_{>i} \text{ for all } i < \omega.$$

Note that if $(a_i : i < \omega) \models p^{(\omega^*)_A}$, then $a_i \mathop{\smile\!\!\!\!\!\!|}_A B a_{>i}$ and $(a_i : i < \omega)$ is B-indiscernible. By assumption each $p^{(n)_A} \restriction A$ is stationary. Then $p^{(-n)_A} \restriction A$ is also stationary and it follows that $r(x_i : i < \omega) = p^{(\omega^*)_A} \restriction A$ is stationary. Its unique global nonforking extension is $\mathfrak{p}^{(\omega^*)}$.

We claim that every realization of $r(x_i : i < \omega)$ (with the increasing order of ω) is A-special. Let $a = (a_i : i < \omega)$ be such a realization an assume $b = (b_i : i < \omega) \equiv_A a$. Choose $c \models p$. By assumption $\mathrm{tp}(ab/A)$ does not fork over A and hence there are $a'b' \equiv_A ab$ such that $a'b' \mathop{\smile\!\!\!\!\!\!|}_A Bc$. Since r is stationary, $\mathrm{tp}(a'/Bc) = \mathfrak{p}^{(\omega^*)} \restriction Bc$. If $i_1 < \ldots < i_n < \omega$, then $(a'_{i_n}, \ldots, a'_{i_1}) \models p^{(n)_A}|_A Bc$ and hence $(c, a'_{i_n}, \ldots, a'_{i_1}) \models p^{(n+1)_A}$. Therefore $a' ^\frown (c)$ is B-indiscernible (if a' is considered a decreasing sequence with order type ω^* we would say $(c) ^\frown a'$ is indiscernible). Similarly, $b' ^\frown (c)$ is B-indiscernible. Now choose c' such that $abc' \equiv_A a'b'c$. Clearly, $a ^\frown (c')$ and $b ^\frown (c')$ are A-indiscernible.

By Corollary 13.12 there is a global type $\mathfrak{q}$ that does not split over A and $a_i \models \mathfrak{q} \restriction A a_{<i}$ for all $i < \omega$. By assumption $\mathfrak{p} \restriction A$ is stationary and hence $\mathfrak{p} = \mathfrak{q}$. Let $a = (a_i : i < \omega) \models p^{(\omega^*)_A}$ and let $c \models \mathfrak{p} \restriction Ba$. Since $\mathfrak{p}$ does not split over B, and $a_i \models \mathfrak{p} \restriction B a_{>i}$, and $c \models \mathfrak{p} \restriction Ba$, then by Remark 7.3 the sequence $(\ldots, a_1, a_0, c)$ is B-indiscernible, that

[4] Suggested by Anand Pillay

is, $(c)^\frown a = (c, a_0, a_1, \ldots)$ is B-indiscernible. Since $\mathfrak{q}$ does not split over B and $a_i \models \mathfrak{q} \restriction Ba_{<i}$ and $c \models \mathfrak{p} \restriction Ba$, again by Remark 7.3 the sequence $a^\frown(c) = (a_0, a_1, \ldots, c)$ is B-indiscernible. Hence $a^\frown(c)$ and $(c)^\frown a$ are B-indiscernible. It follows that $a_0a_1 \equiv_B a_1a_0$. Thus $(p \otimes_A p)^{-1} = \mathrm{tp}(a_0a_1/B) = \mathrm{tp}(a_1a_0/B) = p \otimes_A p$. □

Theorem 15.14 *Assume T has* NIP *and $\mathfrak{p}$ is A-invariant. The following are equivalent:*

1. *$\mathfrak{p}$ is generically stable over A.*
2. *$\mathfrak{p} = \mathrm{Av}(a/\mathfrak{C})$ for every Morley sequence $a = (a_i : i < \omega)$ over any $M \supseteq A$ with global type $\mathfrak{p}$.*
3. *$(\mathfrak{p} \otimes \mathfrak{q})^{-1} = \mathfrak{q} \otimes \mathfrak{p}$ for all B-invariant $\mathfrak{q}$, for all B.*
4. *$(\mathfrak{p} \otimes \mathfrak{p})^{-1} = \mathfrak{p} \otimes \mathfrak{p}$*

Proof: *3* $\Rightarrow$ *4* is obvious and *4* $\Rightarrow$ *1* follows from Lemma 15.12 and Theorem 15.8. *1* $\Rightarrow$ *2* follows from Proposition 15.4 and Theorem 15.8.

2 $\Rightarrow$ *3*. Let $\mathfrak{q}$ be B-invariant. Let $\varphi(x, y) \in \mathfrak{p} \otimes \mathfrak{q}$. We will check that $\varphi^{-1}(y, x) \in \mathfrak{q} \otimes \mathfrak{p}$. Choose a model $M \supseteq AB$ complete over AB and such that $\varphi(x, y) \in L(M)$. Choose a Morley sequence $a = (a_i : i < \omega)$ over M with global type $\mathfrak{p}$. By *2* $\mathfrak{p} = \mathrm{Av}(a/\mathfrak{C})$. Let $b \models \mathfrak{q} \restriction Ma$. Then

$$(a_i, b) \models \mathfrak{p} \restriction M \otimes_B \mathfrak{q} \restriction M = (\mathfrak{p} \otimes \mathfrak{q}) \restriction M \text{ for all } i < \omega.$$

In particular $\models \varphi(a_i, b)$ for all $i < \omega$. This implies $\varphi(x, b) \in \mathrm{Av}(a/\mathfrak{C}) = \mathfrak{p}$. Choose now $c \models \mathfrak{p} \restriction Mb$. Then

$$(b, c) \models \mathfrak{q} \restriction M \otimes_A \mathfrak{p} \restriction M = (\mathfrak{q} \otimes \mathfrak{p}) \restriction M$$

and $\models \varphi(c, b)$, that is $\models \varphi^{-1}(b, c)$. Therefore $\varphi^{-1}(y, x) \in \mathfrak{q} \otimes \mathfrak{p}$. □

16 Extension bases

Definition 16.1 A set A is an *extension base* if no type over A forks over A. In other terms, every $p(x) \in S(A)$ has a global nonforking extension.

Lemma 16.2 *If $\mathfrak{p}$ is A-invariant, then $\mathfrak{p} \restriction A \vdash \mathfrak{p} \restriction \mathrm{bdd}(A)$.*

Proof: Choose $a \models \mathfrak{p} \restriction \mathrm{bdd}(A)$. Assume now $b \models \mathfrak{p} \restriction A$. We claim that $a \equiv_{\mathrm{bdd}(A)} b$. Since $a \equiv_A b$, there is some $f \in \mathrm{Aut}(\mathfrak{C}/A)$ such that $f(a) = b$. Note that f fixes setwise $\mathrm{bdd}(A)$. By A-invariance $\mathfrak{p}^f = \mathfrak{p}$. Hence $b = f(a) \models (\mathfrak{p} \restriction \mathrm{bdd}(A))^f = \mathfrak{p} \restriction \mathrm{bdd}(A)$. □

Lemma 16.3 *If for every finite subtuple a' of a there is a global A-invariant extension of $\mathrm{tp}(a'/A)$, then there is also a global A-invariant extension of $\mathrm{tp}(a/A)$.*

Proof: By compactness, since if $p(x) = \mathrm{tp}(a/A)$ it is enough to prove the consistency of

$$p(x) \cup \{\varphi(x, b) \leftrightarrow \varphi(x, c) : b \equiv_A c \text{ and } \varphi(x, y) \in L(A)\}.$$

□

Lemma 16.4 *Assume T has* NIP*, $A = \mathrm{acl}^{\mathrm{eq}}(A)$ and $e \in \mathrm{acl}^{\mathrm{eq}}(Aa)$. If $\mathrm{tp}(a/A)$ has a global A-invariant extension, then $\mathrm{tp}(ae/A)$ has a global A-invariant extension too.*

Proof: Let $p(x) = \mathrm{tp}(a/A)$ and let $\mathfrak{p} \supseteq p$ be a global A-invariant extension. Let $q(x, y) = \mathrm{tp}(ae/A)$ and choose $\mathfrak{q} \supseteq q$, a global type such that $\mathfrak{q} \restriction x = \mathfrak{p}$. We claim that $\mathfrak{q}$ is A-invariant. Choose $\delta(x, y) \in q$ such that for some $m < \omega$, $\delta(x, y) \vdash \exists^{\leq m} y \delta(x, y)$.

We will prove that $\mathfrak{q}$ is A-invariant applying Lemma 11.12. It is enough to check that $\mathfrak{q}$ does not fork over A and that for each $n < \omega$, $\mathfrak{q}^{(n)} \restriction A$ is a Lascar strong type.

We first claim that $\mathfrak{q}$ does not fork over A. In order to check this, let $\varphi(x, y; z) \in L(A)$ and assume that $\varphi(x, y; b) \in \mathfrak{q}$ divides over A. There is an A-indiscernible sequence $(b_i : i < \omega)$ with $b = b_0$ such that $\{\varphi(x, y; b_i) : i < \omega\}$ is inconsistent. Without loss of generality, $\varphi(x, y; z) \vdash \delta(x, y)$. Since $\mathfrak{p}$ does not fork over A, there is some a' such that $\models \exists y \varphi(a', y; b_i)$ for all $i < \omega$. For each $i < \omega$, choose e_i such that $\models \varphi(a', e_i, b_i)$. Since $\models \delta(a', e_i)$, for some infinite $I \subseteq \omega$, for all $i, j \in I$, $e_i = e_j$. Therefore if $j \in I$, then $\models \varphi(a', e_j, b_i)$ for all $i \in I$. By indiscernibility over A, $\{\varphi(x, y; b_i) : i < \omega\}$ is consistent, a contradiction.

Let $n < \omega$. Since $\mathfrak{p}^{(n)}$ is A-invariant, by Lemma 16.2 and Corollary 11.9, $\mathfrak{p}^{(n)} \restriction A$ is a Lascar-strong type. We claim that $\mathfrak{q}^{(n)} \restriction A$ is a Lascar strong type too. To begin with, we claim it gives rise only to finitely many Lascar strong types over A. Assume that, on the contrary,

$$\{((a_1^i, e_1^i), \ldots, (a_n^i, e_n^i)) : i < \omega\}$$

are realizations of $\mathfrak{q}^{(n)} \restriction A$ with different Lascar strong type over A. Since $a_1^i, \ldots, a_n^i \overset{\mathrm{Ls}}{\equiv}_A a_1^j, \ldots, a_n^j$, we may assume $a_1^i, \ldots, a_n^i = a_1^j, \ldots, a_n^j = a_1, \ldots, a_n$ for all i, j. Since $e_i^j \equiv_{Aa_i} e_i^k$ for all i, k and $e_i^j \in \mathrm{acl}^{\mathrm{eq}}(Aa_i)$, by Ramsey's Theorem there is some infinite $I \subseteq \omega$ such that $e_i^j = e_i^k$ for all $j, k \in I$ for all $i = 1, \ldots, n$. Then $e_i^j = e_i^k$ for all $j, k \in I$ for all $i = 1, \ldots, n$.

Thus, $\overset{\mathrm{Ls}}{\equiv}_A$ has only finitely many classes on $\mathfrak{q}^{(n)} \restriction A$. Since $\mathfrak{q}^{(n)}$ does not fork over A, by Corollary 11.9 $\overset{\mathrm{Ls}}{\equiv}_A = \overset{\mathrm{KP}}{\equiv}_A$ on $\mathfrak{q}^{(n)} \restriction A$ is a bounded A-type-definable equivalence relation E. Let $b_1, \ldots, b_m$ be representatives of the different E-classes and for each two different $i, j \leq m$ choose a formula $\varphi_{ij}(x, y) \in E(x, y)$ such that $\models \neg\varphi_{ij}(a_i, a_j)$ and choose then some $\psi_{ij}(x, y) \in E(x, y)$ such that

$$\psi_{ij}(x, y) \wedge \psi_{ij}(y, z) \wedge \psi_{ij}(z, u) \vdash \varphi_{ij}(x, u).$$

It is easy to check that $\psi(x, y) = \bigwedge_{i<j\leq m} \psi_{ij}(x, y)$ defines E on $\mathfrak{q}^{(n)} \restriction A$. We may assume $\psi(x, y) \in L(A)$ defines an equivalence relation F with finitely many classes in the whole universe. Each F-class is interdefinable over A with some element of $\mathrm{acl}^{\mathrm{eq}}(A)$. Since $A = \mathrm{acl}^{\mathrm{eq}}(A)$, this implies that F (and hence also $\overset{\mathrm{Ls}}{\equiv}_A$) has only one class in $\mathfrak{q}^{(n)} \restriction A$. $\square$

Proposition 16.5 *Assume T has* NIP. *The following are equivalent:*

1. *Every set A is an extension base and $\overset{\mathrm{Ls}}{\equiv}_A = \overset{\mathrm{s}}{\equiv}_A$.*
2. *For any $A = \mathrm{acl}^{\mathrm{eq}}(A)$, every $p(x) \in S_1(A)$ (in the home sort) has a global A-invariant extension.*
3. *For any $A = \mathrm{acl}^{\mathrm{eq}}(A)$, every $p(x) \in S(A)$ has a global A-invariant extension.*

Proof: *1 ⇒ 2.* Let $p(x) \in S_1(A)$, where $A = \mathrm{acl}^{\mathrm{eq}}(A)$. Since A is an extension base, there is a nonforking extension $\mathfrak{p}$ of p. Then $\mathfrak{p}$ does not Lascar-split over A. Since $A = \mathrm{acl}^{\mathrm{eq}}(A)$ and $\overset{\mathrm{s}}{\equiv}_A = \overset{\mathrm{Ls}}{\equiv}_A$, $\mathfrak{p}$ does not split over A, that is, $\mathfrak{p}$ is A-invariant.

3 ⇒ 1. Let $p(x) \in S(A)$ and let $q(x) \in S(\mathrm{acl}^{\mathrm{eq}}(A))$ be some extension of p. By *3* there is an $\mathrm{acl}^{\mathrm{eq}}(A)$-invariant global extension $\mathfrak{p}$ of q. Since $\mathfrak{p}$ does not fork over $\mathrm{acl}^{\mathrm{eq}}(A)$, it does not fork over A. This shows that A is an extension base. Now assume $a \overset{\mathrm{s}}{\equiv}_A b$, that is $a \equiv_{\mathrm{acl}^{\mathrm{eq}}(A)} b$. By *3* applied to $\mathrm{acl}^{\mathrm{eq}}(A)$ and Lemma 16.2, $a \equiv_{\mathrm{bdd}(A)} b$, that is $a \overset{\mathrm{KP}}{\equiv}_A b$. By Corollary 11.9, $a \overset{\mathrm{Ls}}{\equiv}_A b$.

2 $\Rightarrow$ *3*. By Lemma 16.3 it is enough to prove the result for finitary types $p(x)$, and this can be done by induction on the length n of x. Assume the result holds for types in n variables and let $p(x_1, \ldots, x_{n+1}) \in S(A)$. Let $M \supseteq A$ be a model complete over A and strongly ω-homogeneous over A. Let $(a_1, \ldots, a_{n+1}) \models p$. Let e be a sequence of imaginaries enumerating $\mathrm{acl}^{\mathrm{eq}}(Aa_1, \ldots, a_n)$. By Lemmas 16.3 and 16.4 there is some e' such that $\mathrm{tp}(e'/M)$ does not split over A and extends $\mathrm{tp}(e/A)$. There are $a'_1, \ldots, a'_n$ such that $a_1, \ldots, a_n, e \equiv_A a'_1, \ldots, a'_n, e'$. Choose B complete over Me'. Since $e' = \mathrm{acl}^{\mathrm{eq}}(e')$, by *2* and conjugation over A, there is some a'_{n+1} such that $a'_{n+1}e' \equiv_A a_{n+1}e$ and $\mathrm{tp}(a'_{n+1}/B)$ does not split over e'.

We claim that $q(x, y) = \mathrm{tp}(a'_{n+1}e'/M)$ does not split over A. To check this, consider some $\varphi(x, y; z) \in L(A)$ and some finite tuple $b \in M$. Since $\mathrm{tp}(e'/M)$ does not split over A and M is strongly ω-homogeneous over A, for each $\psi(y, z) \in L$, the set $\{b' \in M :\models \psi(e', b')\}$ is invariant under $\mathrm{Aut}(M/A)$. Hence also $\{b' \in M :\models \psi(e', b') \leftrightarrow \psi(e', b)\}$ and $\{b' \in M : e'b' \equiv e'b\}$ are invariant under $\mathrm{Aut}(M/A)$. Since $\mathrm{tp}(a'_{n+1}/B)$ does not split over e', for all $b' \in M$, $e'b' \equiv e'b$ implies $\varphi(x, y; b) \in q \Leftrightarrow \varphi(x, y; b') \in q$. If $b' \in M$ and $b \equiv_A b'$, then $f(b) = b'$ for some $f \in \mathrm{Aut}(M/A)$. Hence $e'b \equiv e'b'$ and therefore $\varphi(x, y; b) \in q \Leftrightarrow \varphi(x, y; b') \in q$.

In particular, $\mathrm{tp}(a'_1, \ldots, a'_{n+1}/M)$ does not split over A and it has a global A-invariant extension. Since $a_1, \ldots, a_{n+1} \equiv_A a'_1, \ldots, a'_{n+1}$, $p(x_1, \ldots, x_{n+1})$ has a global A-invariant extension too. □

17 Abstract preindependence and independence relations

Definition 17.1 Let $\mathop{\smile\!\!\!\!\!\!|}$ be a ternary relation between sets. We consider a list of possible properties of $\mathop{\smile\!\!\!\!\!\!|}$:

Invariance: If $A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $f \in \mathrm{Aut}(\mathfrak{C})$, then $f(A) \mathop{\smile\!\!\!\!\!\!|}_{f(C)} f(B)$.

Monotonicity: If $A \mathop{\smile\!\!\!\!\!\!|}_C B$, $A' \subseteq A$, and $B' \subseteq B$, then $A' \mathop{\smile\!\!\!\!\!\!|}_C B'$.

Right base monotonicity: If $A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $C \subseteq D \subseteq B$, then $A \mathop{\smile\!\!\!\!\!\!|}_D B$.

Left base monotonicity: If $A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $C \subseteq D \subseteq A$, then $A \mathop{\smile\!\!\!\!\!\!|}_D B$.

Right normality: If $A \mathop{\smile\!\!\!\!\!\!|}_C B$, then $A \mathop{\smile\!\!\!\!\!\!|}_C CB$.

Left normality: If $A \mathop{\smile\!\!\!\!\!\!|}_C B$, then $AC \mathop{\smile\!\!\!\!\!\!|}_C B$.

Right transitivity: If $C \subseteq B \subseteq D$, $A \mathop{\smile\!\!\!\!\!\!|}_C B$, and $A \mathop{\smile\!\!\!\!\!\!|}_B D$, then $A \mathop{\smile\!\!\!\!\!\!|}_C D$.

Left transitivity: If $C \subseteq B \subseteq D$, $B \mathop{\smile\!\!\!\!\!\!|}_C A$, and $D \mathop{\smile\!\!\!\!\!\!|}_B A$, then $D \mathop{\smile\!\!\!\!\!\!|}_C A$.

Symmetry: If $A \mathop{\smile\!\!\!\!\!\!|}_C B$, then $B \mathop{\smile\!\!\!\!\!\!|}_C A$.

Right finite character: If $A \mathop{\smile\!\!\!\!\!\!|}_C B_0$ for all finite $B_0 \subseteq B$, then $A \mathop{\smile\!\!\!\!\!\!|}_C B$.

Left finite character: If $A_0 \mathop{\smile\!\!\!\!\!\!|}_C B$ for all finite $A_0 \subseteq A$, then $A \mathop{\smile\!\!\!\!\!\!|}_C B$.

Strong finite character: If $A \not\downfall_C B$, then there are finite tuples $a \in A$, $b \in B$ and some formula $\varphi(x, y) \in L(C)$ such that $\models \varphi(a, b)$ and such that $a' \not\downfall_C b$ for all $a' \models \varphi(x, b)$.

Local character: For every A there is a cardinal number $\kappa(A)$ such that for any B there is some $C \subseteq B$ such that $|C| < \kappa(A)$ and $A \downfall_C B$.

(Right) extension: If $A \downfall_C B$ and $B' \supseteq B$, then $A' \downfall_C B'$ for some $A' \equiv_{BC} A$.

Left extension: If $A \downfall_C B$ and $A' \supseteq A$, then $A' \downfall_C B'$ for some $B' \equiv_{AC} B$.

Anti-reflexivity: If $A \downfall_C A$, then $A \subseteq \mathrm{acl}(C)$.

Right algebraicity: If $A \downfall_C B$, then $A \downfall_C \mathrm{acl}(B)$.

Left algebraicity: If $A \downfall_C B$, then $\mathrm{acl}(A) \downfall_C B$.

Base algebraicity: If $A \downfall_C B$, then $A \downfall_{\mathrm{acl}(C)} B$.

Existence: $A \downfall_C C$.

Let us call *basic axioms* to invariance, monotonicity, right base monotonicity, left transitivity, and left normality. A *preindependence relation* is a ternary relation $\downfall$ satisfying the basic axioms and strong finite character. An *independence relation* is a ternary relation $\downfall$ satisfying the basic axioms, left finite character, local character and extension. Note that invariance and extension imply right normality. Hence all independence relations satisfy right normality.

If $A \downfall^1_C B$ implies $A \downfall^2_C B$ for all A, B, C we say that $\downfall^1$ is *stronger* than $\downfall^2$ and that $\downfall^2$ is *weaker* than $\downfall^1$.

Fact 17.2 *All independence relations are symmetric.*

Proof: See [1], or [5], or [7]. □

Definition 17.3 Given a ternary relation $\downfall$, we define the ternary relation $\downfall^*$ as follows:

$$A \mathop{\downfall}^{*}_{C} B \text{ if and only if for all } B' \supseteq B \text{ there is some } A' \equiv_{BC} A \text{ such that } A' \mathop{\downfall}_{C} B'$$

Proposition 17.4 *1. If $\downfall$ is invariant, then $\downfall^*$ is invariant and stronger than $\downfall$.*

2. *If $\downfall$ satisfies monotonicity and invariance, then $\downfall^*$ satisfies extension.*

3. *Each basic axiom and also anti-reflexivity transfers from $\downfall$ to $\downfall^*$.*

4. *Assume $\downfall$ satisfies the basic axioms and left finite character. If $\downfall^*$ satisfies local character, then it is an independence relation.*

5. *Assume $\downfall$ satisfies monotonicity and invariance. Then, $\downfall = \downfall^*$ if and only if $\downfall$ satisfies extension.*

6. *Assume $\downfall$ is invariant and satisfies monotonicity and strong finite character. Then $\downfall^*$ satisfies strong finite character.*

7. *If $\downfall$ satisfies left algebraicity, then $\downfall^*$ satisfies left algebraicity too.*

8. *If $\mathop{\smile\!\!\!\!\!\!|}$ satisfies monotonicity and invariance, $\mathop{\smile\!\!\!\!\!\!|}^*$ satisfies right algebraicity. If moreover $\mathop{\smile\!\!\!\!\!\!|}$ satisfies right base monotonicity, $\mathop{\smile\!\!\!\!\!\!|}^*$ satisfies also base algebraicity.*

Proof: Everything (except *7* and *8*) is from Adler [1]. Points *1–5* are also proved in [7]. *7* is straightforward. We prove now *6*. Assume $\mathop{\smile\!\!\!\!\!\!|}$ has strong finite character and $a \not\mathop{\smile\!\!\!\!\!\!|}^*_C B$. For some $B' \supseteq B$, for all $a' \equiv_{BC} a$, $a' \not\mathop{\smile\!\!\!\!\!\!|}_C B'$. Let $p(x) = \mathrm{tp}(a/BC)$ and let

$$\pi(x) = \{\neg\varphi(x,b) : b \in B',\ \varphi(x,y) \in L(C),\ \text{and } a' \not\mathop{\smile\!\!\!\!\!\!|}_C b \text{ for all } a' \models \varphi(x,b)\}$$

By strong finite character of $\mathop{\smile\!\!\!\!\!\!|}$, $\pi(x) \cup p(x)$ is inconsistent and therefore for some $\psi(x,y) \in L(C)$, for some $b \in B$, $\psi(x,b) \in p(x)$ and $\pi(x) \cup \{\psi(x,b)\}$ is inconsistent. Note that $\models \psi(a,b)$. We claim that for all $a' \models \psi(x,b)$, $a' \not\mathop{\smile\!\!\!\!\!\!|}^*_C b$. To check this, assume $\models \psi(a',b)$ but $a' \mathop{\smile\!\!\!\!\!\!|}^*_C b$. By definition of $\mathop{\smile\!\!\!\!\!\!|}^*$, there is some $a'' \equiv_{Cb} a'$ such that $a'' \mathop{\smile\!\!\!\!\!\!|}_C B'$. Then $\models \psi(a'',b)$ and $a'' \models \pi(x)$, a contradiction.

8. Assume $a \mathop{\smile\!\!\!\!\!\!|}^*_C B$. By extension, there is some $a' \equiv_{BC} a$ such that $a' \mathop{\smile\!\!\!\!\!\!|}^*_C \mathrm{acl}(B)$. Fix some $f \in \mathrm{Aut}(\mathfrak{C}/BC)$ such that $f(a') = a$. Since $f(\mathrm{acl}(BC)) = \mathrm{acl}(BC)$, by invariance $a \mathop{\smile\!\!\!\!\!\!|}^*_C \mathrm{acl}(BC)$. By monotonicity $a \mathop{\smile\!\!\!\!\!\!|}^*_C \mathrm{acl}(B)$. On the other hand, by monotonicity and right base monotonicity, we conclude $A \mathop{\smile\!\!\!\!\!\!|}^*_{\mathrm{acl}(C)} B$. □

Proposition 17.5 *If $\mathop{\smile\!\!\!\!\!\!|}$ is a preindependence relation, then $\mathop{\smile\!\!\!\!\!\!|}^*$ is the weakest preindependence relation that satisfies extension and is stronger than $\mathop{\smile\!\!\!\!\!\!|}$.*

Proof: By points *1*, *2*, *3*, and *6* of Proposition 17.4, we know that $\mathop{\smile\!\!\!\!\!\!|}^*$ is a preindependence relation and satisfies extension. Note that if $\mathop{\smile\!\!\!\!\!\!|}_1$ is stronger than $\mathop{\smile\!\!\!\!\!\!|}_2$, then $\mathop{\smile\!\!\!\!\!\!|}^*_1$ is stronger than $\mathop{\smile\!\!\!\!\!\!|}^*_2$. Now assume $\mathop{\smile\!\!\!\!\!\!|}_1$ is a preindependence relation with extension and it is stronger than $\mathop{\smile\!\!\!\!\!\!|}$. By point *5* of proposition 17.4, $\mathop{\smile\!\!\!\!\!\!|}_1 = \mathop{\smile\!\!\!\!\!\!|}^*_1$ is stronger than $\mathop{\smile\!\!\!\!\!\!|}^*$. □

Definition 17.6 $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ will denote nonforking independence: $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_C B$ if and only if for every tuple $a \in A$, $\mathrm{tp}(a/BC)$ does not fork over C. In the previous chapters we have used $\mathop{\smile\!\!\!\!\!\!|}$ for this relation, but now $\mathop{\smile\!\!\!\!\!\!|}$ is used as an arbitrary ternary relation on sets. Nondividing independence can be defined similarly: $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_C B$ if and only if $\mathrm{tp}(a/BC)$ does not divide over C for all tuples $a \in A$.

Fact 17.7 *$A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_C B$ if and only if for any C-indiscernible sequence $(b_i : i < \omega)$ with $b_0 \in BC$ there is some AC-indiscernible sequence $(b'_i : i < \omega) \equiv_{b_0 C} (b_i : i < \omega)$.*

Proof: See, for instance, Chapter 4 of [7]. □

Remark 17.8 *1.* *$\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}$ is a preindependence relation and moreover it satisfies anti-reflexivity, right normality, existence, and left algebraicity.*

2. $(\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}})^* = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$.

3. *$\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ is a preindependence relation and moreover it satisfies anti-reflexivity, right normality, extension, and all algebraicity conditions.*

Proof: *1.* Left algebraicity of $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}$ follows from the fact that $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_C B \Rightarrow \mathrm{acl}(AC) \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_C B$, which can be easily checked using Fact 17.7.

2 is Proposition 12.14 of [7].

3. By Proposition 17.5 $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ is a preindependence relation and satisfies extension. The remaining points follow from Proposition 17.4. □

Fact 17.9 *If $\mathop{\smile\!\!\!\!\!\!\mid}$ is an independence relation, then $A \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}_C B \Rightarrow A \mathop{\smile\!\!\!\!\!\!\mid}_C B$.*

Proof: See Proposition 12.19 of [7]. □

Fact 17.10 *If T is simple, then $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}} = \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}$. Moreover, the following are equivalent:*

1. *T is simple.*
2. *$\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}$ is an independence relation.*
3. *$\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}$ is an independence relation.*
4. *$\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}$ satisfies local character.*
5. *$\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}$ satisfies local character.*
6. *$\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}$ is symmetric.*
7. *$\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}$ is symmetric.*
8. *$\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}$ is right transitive.*
9. *$\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}$ is right transitive.*

Proof: See propositions 12.16 and 12.24 of [7]. □

Fact 17.11 *T is simple if and only if in T there is an independence relation $\mathop{\smile\!\!\!\!\!\!\mid}$ which satisfies the independence theorem over models: for any model M for any $A, B \supseteq M$ such that $A \mathop{\smile\!\!\!\!\!\!\mid}_M B$, if $a \mathop{\smile\!\!\!\!\!\!\mid}_M A$, and $b \mathop{\smile\!\!\!\!\!\!\mid}_M B$ and $a \equiv_M b$, then there is some c such that $c \mathop{\smile\!\!\!\!\!\!\mid}_M AB$, $c \equiv_A a$ and $c \equiv_B b$. Moreover, if T is simple and $\mathop{\smile\!\!\!\!\!\!\mid}$ is as indicated, then $\mathop{\smile\!\!\!\!\!\!\mid} = \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}$.*

Proof: See Theorem 12.21 of [7]. □

18 More preindependence relations

Definition 18.1

1. $A \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{u}}_C B$ if and only if $\mathrm{tp}(a/BC)$ is finitely satisfiable in C for all tuples $a \in A$.
2. $A \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{s}}_C B$ if and only if for all tuples $b_1, b_2 \in BC$, if $\models \mathrm{nc}_C(b_1, b_2)$, then $\models \mathrm{nc}_{AC}(b_1, b_2)$.
3. $A \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}_C B$ if and only if for each tuple $a \in A$ there is a global extension $\mathfrak{p}$ of $\mathrm{tp}(a/BC)$ that does not Lascar-split over C.

Proposition 18.2

1. *$\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{u}}$ is a preindependence relation. Moreover it satisfies right normality and anti-reflexivity.*
2. *$\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{u}}$ satisfies extension. Hence $(\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{u}})^* = \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{u}}$ and it satisfies right and base algebraicity.*
3. *$A \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{u}}_C B$ if and only if for every tuple $a \in A$ there is a sequence $b = (b_i : i \in I)$ in C and some ultrafilter U on I such that $\mathrm{tp}(a/BC) = \lim_U(b/BC)$.*

Proof: *1* is clear.

2. By compactness, every type $p(x) \in S(BC)$ finitely satisfiable in C can be extended to a complete type over BCD finitely satisfiable in C.

3. Every $p(x) \in S(BC)$ finitely satisfiable in C is in fact $\lim_U(b/BC)$ for some sequence b of tuples in C, for some ultrafilter U on $I = p(x)$: the ultrafilter extends the set of all $[\varphi] = \{\psi \in p : \psi \equiv \varphi\}$ with $\varphi \in p$ and the sequence is obtained by choosing some $b_\varphi \models \varphi$ in C for every $\varphi \in p$. $\square$

Remark 18.3 *Notice that $A \not\mathop{\smile\!\!\!\!\!\mid}^{u} B$ for all $A \neq \emptyset$. In stable T, $\mathop{\smile\!\!\!\!\!\mid}^{u}_M = \mathop{\smile\!\!\!\!\!\mid}^{f}_M$ for every model M. In simple unstable T, $\mathop{\smile\!\!\!\!\!\mid}^{u}_M \neq \mathop{\smile\!\!\!\!\!\mid}^{f}_M$ for some model M.*

Proposition 18.4 *$\mathop{\smile\!\!\!\!\!\mid}^{s}$ is a preindependence relation and satisfies right normality, left base monotonicity and left and base algebraicity.*

Proof: Invariance, monotonicity, left and right normality, and left transitivity are straightforward.

Right base monotonicity. Assume $A \mathop{\smile\!\!\!\!\!\mid}^{s}_C B$, and $C \subseteq D \subseteq B$, and let us show that $A \mathop{\smile\!\!\!\!\!\mid}^{s}_D B$. Let $b_1, b_2 \in BD = BC$ be such that $\models \mathrm{nc}_D(b_1, b_2)$ and let d enumerate D. Then $\models \mathrm{nc}_C(b_1 d, b_2 d)$ and therefore $\models \mathrm{nc}_{AC}(b_1 d, b_2 d)$. It follows that $\models \mathrm{nc}_{AD}(b_1, b_2)$.

Strong finite character. Let $A \not\mathop{\smile\!\!\!\!\!\mid}^{s}_C B$ and let $b_1, b_2 \in BC$ be such $\models \mathrm{nc}_C(b_1, b_2)$ and $\not\models \mathrm{nc}_{AC}(b_1, b_2)$. For some tuple $a \in A$, for some $\theta(x, y, z) \in L(C)$, $\theta(x, y, a) \in \mathrm{nc}_{AC}(x, y)$ and $\not\models \theta(b_1, b_2, a)$. We may assume that for every a', $\theta(x, y, a')$ is a thick formula. If $\models \neg\theta(b_1, b_2, a')$, then $a' \not\mathop{\smile\!\!\!\!\!\mid}^{s}_C b_1 b_2$, because $\theta(x, y, a') \in \mathrm{nc}_{Ca'}(x, y)$.

Left base monotonicity: clear, since in the definition of $\mathop{\smile\!\!\!\!\!\mid}^{s}$ we may always assume that $b_1, b_2 \in B$.

Finally, it is clear that $A \mathop{\smile\!\!\!\!\!\mid}^{s}_C B \Rightarrow \mathrm{acl}(AC) \mathop{\smile\!\!\!\!\!\mid}^{s}_C B$, and this implies left and base algebraicity. $\square$

Proposition 18.5 *If $M \supseteq C$ is ω-saturated over C, then the following are equivalent:*

1. *$A \mathop{\smile\!\!\!\!\!\mid}^{s}_C M$*
2. *$\mathrm{tp}(a/M)$ does not strongly split over C for all tuples $a \in A$.*
3. *$\mathrm{tp}(a/M)$ does not Lascar-split over C for all tuples $a \in A$.*

Proof: *1* $\Rightarrow$ *2* is clear and does not need the assumption of ω-saturation.

3 $\Rightarrow$ *1.* Assume $b_0, b_1 \in M$ and $\models \mathrm{nc}_C(b_0, b_1)$. By ω-saturation over C, there is a C-indiscernible sequence $(b_i : i < \omega)$ in M. We claim that it is AC-indiscernible. Let $a \in A$ be a tuple, let $n < \omega$ and let $i_0 < \ldots < i_n < \omega$. We must check that $b_0, \ldots, b_n \equiv_{aC} b_{i_0}, \ldots, b_{i_n}$. But this is clear, since $b_0, \ldots, b_n \overset{\mathrm{Ls}}{\equiv}_C b_{i_0}, \ldots, b_{i_n}$ and hence $\models \varphi(a, b_0, \ldots, b_n) \leftrightarrow \varphi(a, b_{i_0}, \ldots, b_{i_n})$ for all $\varphi(x, y_0, \ldots, y_n) \in L(C)$.

2 $\Leftrightarrow$ *3.* By Remark 9.8. $\square$

Remark 18.6 *Assume B is Lascar-complete over $C \subseteq B$. The following are equivalent:*

1. *$\mathrm{tp}(a/B)$ does not Lascar-split over C for all tuples $a \in A$.*

2. $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}_C B$.

Proof: See Proposition 10.1. □

Proposition 18.7 $(\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{s}})^* = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}$.

Proof: It is clear that $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}$ is stronger than $(\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{s}})^*$. We prove $A(\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{s}})^*_C B \Rightarrow A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}_C B$. Assume $A(\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{s}})^*_C B$ and choose a model $M \supseteq BC$ Lascar-complete over C and ω-saturated over C. There is some $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{s}}_C M$. By Proposition 18.5 and Remark 18.6, $A' \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}_C M$. It follows that $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}_C B$. □

Corollary 18.8 $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}$ *is a preindependence relation and it satisfies additionally extension, right-normality, anti-reflexivity, and all algebraicity conditions.*

Proof: By propositions 18.7, 18.4, and 17.4. □

Proposition 18.9 $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{u}}_C B \Rightarrow A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}_C B \Rightarrow A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_C B$

Proof: $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{u}}$ has the extension property and a global type finitely satisfiable in C does not split over C. Similarly, a global type does not fork over C if it does not Lascar-split over C. □

Definition 18.10 Let f be a function assigning a cardinal number to each cardinal number. We say that $\mathop{\smile\!\!\!\!\!\!|}$ *is bounded by* f if for all $C \subseteq B$ for every finitary type $p(x) \in S(C)$, there are at most $f(|T| + |C|)$ types $q(x) \in S(B)$ extending p such that for any $a \models q$, $a \mathop{\smile\!\!\!\!\!\!|}_C B$. We say that $\mathop{\smile\!\!\!\!\!\!|}$ *is bounded* if it is bounded by some f.

Proposition 18.11 $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}$ *is the weakest bounded preindependence relation that satisfies the extension axiom, and it is bounded by* $f(\kappa) = 2^{2^{\kappa}}$.

Proof: For any finitary $p(x) \in S(C)$, the number of global types $\mathfrak{p}$ extending p that do not Lascar-split over C is bounded by $2^{2^{|T|+|C|}}$. Hence, $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}$ is bounded by $f(\kappa) = 2^{2^{\kappa}}$. Now let $\mathop{\smile\!\!\!\!\!\!|}$ be a bounded preindependence relation satisfying extension and assume $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}$ is not weaker than $\mathop{\smile\!\!\!\!\!\!|}$. There is a tuple a and sets C, B such that $a \mathop{\smile\!\!\!\!\!\!|}_C B$ and $a \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}_C B$. By extension, we may assume B is a $(|C| + |T|)^+$-saturated model containing C. By Remark 18.6 and Proposition 18.5, $a \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{s}}_C B$. Hence, using saturation of B, there is a C-indiscernible sequence $b = (b_i : i < \omega)$ in B such that for some $\varphi(x, y) \in L(C)$, $\models \varphi(a, b_0)$ and $\models \neg\varphi(a, b_1)$. In fact we obtain $i_1 < \ldots < i_n$ and $\psi(x, y_1, \ldots, y_n) \in L(C)$ such that $\models \psi(a, b_1, \ldots, b_n)$ and $\not\models \psi(a, b_{i_1}, \ldots, b_{i_n})$, but we may then assume $n < i_0$ and we can consider a derived C-indiscernible sequence of n-tuples of b_i's giving the result. We may assume that $\models \varphi(a, b_i)$ for all $i \geq 2$ or $\models \neg\varphi(a, b_i)$ for all $i \geq 2$. Without loss of generality, we may assume we are in the second case. Note that $a \mathop{\smile\!\!\!\!\!\!|}_C b$. Let $c = (c_i : i < \kappa)$ be a C-indiscernible sequence with same Ehrenfeucht-Mostowski set over C as b. We claim that $p(x) = \mathrm{tp}(a/C)$ has at least κ extensions $q_i(x)$ over Cc such that $a' \mathop{\smile\!\!\!\!\!\!|}_C c$ for all $a' \models q_i$. Let $\pi(x)$ be the set of all formulas $\neg\psi(x)$ such that $\psi(x) \in L(Cc)$ an for all $a' \models \psi(x)$, $a' \not\mathop{\smile\!\!\!\!\!\!|}_C c$. For each $i < \kappa$, $\pi(x) \cup p(x) \cup \{\varphi(x, c_i)\} \cup \{\neg\varphi(x, c_j) : j > i\}$ is consistent and we may choose a type $q_i(x) \in S(Cc)$ extending it. If $i < j$, then $q_i \neq q_j$ since $\neg\varphi(x, c_j) \in q_i$ while $\varphi(x, c_j) \in q_j$. If $a' \models q_i$, then, by strong finite character, $a' \mathop{\smile\!\!\!\!\!\!|}_C c$. This contradicts boundedness of $\mathop{\smile\!\!\!\!\!\!|}$. □

Proposition 18.12 *The following are equivalent and they hold if* T *has* NIP.

1. $\mathop{\smile\!\!\!\!\!\mid}^{\text{f}}$ *is bounded.*

2. $\mathop{\smile\!\!\!\!\!\mid}^{\text{f}}$ *is bounded by* $f(\kappa) = 2^{2^{\kappa}}$.

3. $\mathop{\smile\!\!\!\!\!\mid}^{\text{f}} = \mathop{\smile\!\!\!\!\!\mid}^{\text{i}}$.

Proof: Since $\mathop{\smile\!\!\!\!\!\mid}^{\text{i}}$ is stronger than $\mathop{\smile\!\!\!\!\!\mid}^{\text{f}}$, the conditions are equivalent by Proposition 18.11. By Proposition 9.6, *3* holds if T has NIP. □

Proposition 18.13 *The following are equivalent.*

1. *T is stable.*

2. *T is simple and* $\mathop{\smile\!\!\!\!\!\mid}^{\text{i}} = \mathop{\smile\!\!\!\!\!\mid}^{\text{f}}$.

3. $\mathop{\smile\!\!\!\!\!\mid}^{\text{i}}$ *has local character.*

4. $\mathop{\smile\!\!\!\!\!\mid}^{\text{i}}$ *is an independence relation.*

5. $\mathop{\smile\!\!\!\!\!\mid}^{\text{i}}$ *is symmetric.*

Proof: *1* ⇒ *2*. By Proposition 18.12, since stable theories are simple and have NIP.

2 ⇒ *3*. Clear since in a simple theory $\mathop{\smile\!\!\!\!\!\mid}^{\text{f}}$ has local character (for instance, see Proposition 12.16 of [7]).

3 ⇒ *4*. Clear, since by Corollary 18.8 $\mathop{\smile\!\!\!\!\!\mid}^{\text{i}}$ satisfies all the other conditions of independence.

4 ⇒ *5*. By Fact 17.2.

5 ⇒ *3*. Given A, B it is easy to find $C \subseteq A$ of cardinality $\leq |B| + |T|$ such that $A \mathop{\smile\!\!\!\!\!\mid}^{\text{u}}_{C} B$. By Proposition 18.9, $A \mathop{\smile\!\!\!\!\!\mid}^{\text{i}}_{C} B$ and by symmetry $B \mathop{\smile\!\!\!\!\!\mid}_{C} A$.

4 ⇒ *1*. By Proposition 18.11, $\mathop{\smile\!\!\!\!\!\mid}^{\text{i}}$ is bounded. Stable theories are characterized by the existence of a bounded independence relation. See, for instance, Theorem 12.22 of [7]. □

19 Algebraic independence

Definition 19.1 $A \mathop{\smile\!\!\!\!\!\mid}^{\text{a}}_{C} B$ if and only if $\operatorname{acl}(AC) \cap \operatorname{acl}(BC) = \operatorname{acl}(C)$.

Proposition 19.2 $\mathop{\smile\!\!\!\!\!\mid}^{\text{a}}$ *satisfies invariance, symmetry, transitivity, monotonicity, normality, finite character, local character, anti-reflexivity, algebraicity, existence, and extension. It satisfies all conditions of an independence relation except, perhaps, base monotonicity. Moreover it is weaker than* $\mathop{\smile\!\!\!\!\!\mid}^{\text{f}}$.

Proof: Invariance, symmetry, transitivity, monotonicity, normality, finite character, anti-reflexivity, algebraicity and existence are easy to check.

Local character. Given A and B, construct $(C_i : i < \omega)$ and $(D_i : i < \omega)$ as follows. Start with $C_0 = D_0 = \emptyset$. Put $D_{i+1} = \operatorname{acl}(AC_i) \cap \operatorname{acl}(B)$. For each $d \in D_{i+1}$ choose a finite subset $C_d \subseteq B$ such that $d \in \operatorname{acl}(C_d)$ and put $C_{i+1} = \bigcup_{d \in D_{i+1}} C_d$. Then $C = \bigcup_{i<\omega} C_i$ is a subset of B of cardinality $\leq |A| + |T|$ and $A \mathop{\smile\!\!\!\!\!\mid}^{\text{a}}_{C} B$.

Extension. It is enough to prove that for all A, B, C there is some $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{a}}_C B$. Let a be an enumeration of $\mathrm{acl}(AC) \setminus \mathrm{acl}(C)$. It suffices to show that for some $a' \equiv_{\mathrm{acl}(C)} a$, $a' \cap \mathrm{acl}(BC) = \emptyset$ since then we can obtain A' such that $A'a' \equiv_{\mathrm{acl}(C)} Aa$ and it follows that $A' \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{a}}_C B$. To obtain a' it is enough to prove that for any finite subtuple a_0 of a there is some $a'_0 \equiv_{\mathrm{acl}(C)} a_0$ such that $a'_0 \cap \mathrm{acl}(BC) = \emptyset$, and this can be done by P.M. Neumann's Lemma (see the Appendix) since $a_0 \cap \mathrm{acl}(C) = \emptyset$. □

Proposition 19.3 $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{a}}$ *satisfies base monotonicity if and only if the lattice of algebraically closed sets is modular, that is, for all algebraically closed* A, B, C, *if* $C \subseteq B$, *then* $B \cap \mathrm{acl}(AC) = \mathrm{acl}((B \cap A)C)$.

Proof: Notice that if B is closed and contains C, then $\mathrm{acl}(B \cap A)C \subseteq B \cap \mathrm{acl}(AC)$. Now assume base monotonicity. Since $A \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{a}}_{A\cap B} B$, we get $A \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{a}}_{(A\cap B)C} B$ and therefore $\mathrm{acl}(AC) \cap B \subseteq \mathrm{acl}((A \cap B)C)$. For the other direction, we assume modularity, $A \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{a}}_C B$ and $C \subseteq D \subseteq B$. Then $\mathrm{acl}(AD) \cap \mathrm{acl}(B) \subseteq \mathrm{acl}((\mathrm{acl}(B) \cap \mathrm{acl}(A))D) \subseteq \mathrm{acl}(\mathrm{acl}(C)D) = \mathrm{acl}(D)$. Hence $A \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{a}}_D B$. □

Fact 19.4 *If* (Ω, cl) *is a pregeometry, the ternary relation* $\mathop{\smile\!\!\!\!\!\!\;|}^{\dim}$ *is defined on subsets of* Ω *by*

$$A \mathop{\smile\!\!\!\!\!\!\;|}^{\dim}_C B \Leftrightarrow \dim(A_0/C) = \dim(A_0/BC) \text{ for all finite } A_0 \subseteq A.$$

It can also be defined by

$$A \mathop{\smile\!\!\!\!\!\!\;|}^{\dim}_C B \Leftrightarrow \text{ every } X \subseteq A \text{ independent over } C \text{ is independent over } BC.$$

It satisfies symmetry, transitivity, normality, motonicity, base motonicity, finite character, existence, and the following stronger form of local character: If A *is finite, then for each* B *there is a finite* $C \subseteq B$ *such that* $A \mathop{\smile\!\!\!\!\!\!\;|}^{\dim}_C B$. *Moreover* $A \mathop{\smile\!\!\!\!\!\!\;|}^{\dim}_C \mathrm{cl}(C)$ *and the following version of anti-reflexivity holds: if* $A \mathop{\smile\!\!\!\!\!\!\;|}^{\dim}_C A$, *then* $A \subseteq \mathrm{cl}(C)$. *The pregeometry is modular if and only if*

$$A \mathop{\smile\!\!\!\!\!\!\;|}^{\dim}_C B \Leftrightarrow \mathrm{cl}(AC) \cap \mathrm{cl}(BC) = \mathrm{cl}(C).$$

Proof: See [6]. □

Proposition 19.5 *Assume the algebraic closure operator* acl *has the exchange property and therefore defines a pregeometry in the universe. Let* $\mathop{\smile\!\!\!\!\!\!\;|}^{\dim}$ *be*[5] *the relation defined as in Fact 19.4. It satisfies invariance and all the properties stated in Fact 19.4. Moreover* $\mathop{\smile\!\!\!\!\!\!\;|}^{\dim}$ *satisfies extension and strong finite character and hence it is an independence relation and also a preindependence relation. In this situation,* $\mathop{\smile\!\!\!\!\!\!\;|}^{\dim} = \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{a}}$ *if and only the pregeometry is modular.*

Proof: We check the extension property, in fact in a strong sense that implies existence. Let $a = (a_i : i < \alpha)$, and let C, B be sets. We prove inductively that for all $\beta \leq \alpha$ there is some $a'_{<\beta} \equiv_C a_{<\beta}$ such that $a'_{<\beta} \mathop{\smile\!\!\!\!\!\!\;|}^{\dim}_C B$. The limit case is clear since we extend the previous obtained sequences. Assume inductively this is the case for $\beta \leq \alpha$ and let us consider the case of $a_{\leq\beta} = a_{<\beta}a_\beta$. We have $a'_{<\beta} \equiv_C a_{<\beta}$ such that $a'_{<\beta} \mathop{\smile\!\!\!\!\!\!\;|}^{\dim}_C B$. Let b be

[5] The notation is $\mathop{\smile\!\!\!\!\!\!\;|}^{g}$ in [2]

such that $a'_{<\beta}b \equiv_C a_{<\beta}a_\beta$. It is enough to find $b' \equiv_{Ca'_{<\beta}} b$ such that $b' \mathop{\smile\!\!\!\!\!\!|}^{\dim}_{Ca'_{<\beta}} B$. In other terms, we must check that for every element a, for all sets B, C there is some $a' \equiv_C a$, such that $a \mathop{\smile\!\!\!\!\!\!|}^{\dim}_C B$. If $a \in \mathrm{acl}(C)$, we put $a' = a$. If $a \notin \mathrm{acl}(C)$, we may choose some $a' \equiv_C a$ such that $a' \notin \mathrm{acl}(BC)$, and it follows that $a' \mathop{\smile\!\!\!\!\!\!|}^{\dim}_C B$.

Strong finite character. Since $\mathop{\smile\!\!\!\!\!\!|}^{\dim}$ is symmetric, it is enough to check that the reverse of $\mathop{\smile\!\!\!\!\!\!|}^{\dim}$ has the property. Assume $A \not\mathop{\smile\!\!\!\!\!\!|}^{\dim}_C B$. Then for some finite tuples $a \in A$ and $b = b_1, \ldots, b_n \in B$, b is algebraically independent over C but $b_1 \in \mathrm{acl}(C, a, b_2, \ldots, b_n)$. Choose $\varphi(x, y_1, \ldots, y_n) \in L(C)$ such that $\models \varphi(a, b_1, \ldots, b_n)$ and such that $b_1 \in \mathrm{acl}(C, a', b_2, \ldots, b_n)$ for every a' such that $\models \varphi(a', b_1, \ldots, b_n)$. Then $b \not\mathop{\smile\!\!\!\!\!\!|}^{\dim}_C a'$ for each such a'. □

Corollary 19.6 *In any o-minimal theory $\mathop{\smile\!\!\!\!\!\!|}^{\dim}$ is an independence relation and also a preindependence relation. It satisfies anti-reflexivity and all algebraicity conditions.*

20 Appendix

Lemma 20.1 (P.M. Neumann) *Assume the group G acts on Ω and all orbits are of size $\geq \kappa \geq \omega$. If $\Gamma \subseteq \Omega$ is finite and $\Delta \subseteq \Omega$ satisfies $|\Delta|^+ < \kappa$, then there exists some $g \in G$ such that $g\Gamma \cap \Delta = \emptyset$.*

Proof: By induction on $|\Gamma|$. It is obvious if $|\Gamma| = 0$. Assume $|\Gamma| = n + 1$. We can assume $\Gamma \not\subseteq \Delta$ (otherwise choose $g \in G$ with $g\Gamma \not\subseteq \Delta$ and replace Γ by $\Gamma' = g\Gamma$). Fix $\gamma_0 \in \Gamma \setminus \Delta$ and put $\Gamma_0 = \Gamma \setminus \{\gamma_0\}$. Using the induction hypothesis we can construct inductively a sequence $(g_i : i < |\Delta|^+)$ of elements of G such that

$$g_i\Gamma_0 \cap (\Delta \cup \bigcup_{j<i} g_j\Delta) = \emptyset$$

for all $i < |\Delta|^+$. Note that $|\bigcup_{i<|\Delta|^+} g_i\Delta| \leq |\Delta|^+ < \kappa$. There are two cases. The first one consists in that $g_i\gamma_0 \notin \Delta$ for some $i < |\Delta|^+$. Then $g_i\Gamma \cap \Delta = \emptyset$. In the second case we have $g_i\gamma_0 \in \Delta$ for all $i < |\Delta|^+$. By cardinality reasons, $g_i\gamma_0 = g_j\gamma_0$ for some $j < i < |\Delta|^+$. Let $g = g_j^{-1}g_i$. Note that $g\gamma_0 = \gamma_0$. Then $g_i\Gamma_0 \cap g_j\Delta = \emptyset$ and therefore $g\Gamma_0 \cap \Delta = \emptyset$. Hence $g\Gamma \cap \Delta = \emptyset$. □

Corollary 20.2 *Let $a_1, \ldots, a_n$ be elements of the monster model such that $a_i \notin \mathrm{acl}(A)$ for all $i = 1, \ldots, n$. For any set B there are $b_1, \ldots, b_n$ such that $b_1 \ldots b_n \equiv a_1 \ldots a_n$ and $b_i \notin B$ for all $i = 1, \ldots, n$.*

Proof: By Lemma 20.1 with $\Omega = \bigcup_{i=1}^n \{a : a \equiv_A a_i\}$, $\Gamma = \{a_1, \ldots, a_n\}$, $\Delta = B \cap \Omega$, $G = \mathrm{Aut}(\mathfrak{C}/A)$ and $\kappa > |\Delta|^+$. □

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