Pregeometries and minimal types

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1 Pregeometries

Definition 1.1 *Let* Ω *be a set (more generally, a class) and let* cl *be a mapping assigning to each* $X \subseteq \Omega$ *some* $\mathrm{cl}(X) \subseteq \Omega$. *We say that* cl *is a* closure operator *if:*

P1. $X \subseteq \mathrm{cl}(X)$.

P2. If $X \subseteq Y$, *then* $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$.

P3. $\mathrm{cl}(\mathrm{cl}(X)) \subseteq \mathrm{cl}(X)$.

The operator cl *is* finitary *if*

P4. For all $a \in \mathrm{cl}(X)$, *there is some finite* $X_0 \subseteq X$ *such that* $a \in \mathrm{cl}(X_0)$.

A finitary closure operator is a pregeometry on Ω *if additionally the* exchange property *holds:*

P5. If $a, b \in \Omega$ *and* $a \in \mathrm{cl}(Xb) \smallsetminus \mathrm{cl}(X)$, *then* $b \in \mathrm{cl}(Xa)$.

Definition 1.2 *Let* (Ω, cl) *be a closure operator. A subset* $X \subseteq \Omega$ *is called* closed *if* $\mathrm{cl}(X) = X$. *Clearly* $\mathrm{cl}(X)$ *is the smallest closed set containing* X. *The intersection of closed sets is closed because* $\mathrm{cl}(\mathrm{cl}(X) \cap \mathrm{cl}(Y)) \subseteq \mathrm{cl}(\mathrm{cl}(X)) \cap \mathrm{cl}(\mathrm{cl}(Y)) = \mathrm{cl}(X) \cap \mathrm{cl}(Y)$. *The closed sets form a lattice with* $\inf(X, Y) = X \cap Y$ *and* $\sup(X, Y) = \mathrm{cl}(X \cup Y)$

Definition 1.3 *Let* (Ω, cl) *be a pregeometry. We say that* $a \in \Omega$ *is* independent of $X \subseteq \Omega$ *if* $a \notin \mathrm{cl}(X)$. *We say that* $X \subseteq \Omega$ *is* independent *if for all* $a \in X$, a *is independent of* $X \smallsetminus \{a\}$. *A* basis of $X \subseteq \Omega$ *is a maximally independent subset of* X. *We will see that all bases of* X *have the same cardinality, which will be called the* dimension *of* X *and will be denoted by* $\dim(X)$.

Lemma 1.4 *Let* (Ω, cl) *be a pregeometry.*

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1. *$X \subseteq \Omega$ is independent if and only if every finite subset of X is independent.*

2. *If X is independent and $a \notin \mathrm{cl}(X)$, then $X \cup \{a\}$ is independent.*

3. *If $(a_i : i < \alpha)$ is an enumeration of $X \subseteq \Omega$ such that $a_i \notin \mathrm{cl}(\{a_j : j < i\})$ for all $i < \alpha$, then X is independent.*

Proof: *1* is clear since the closure operator is finitary. *2* follows from *P5*. By *1*, to prove *3* it is enough to check that any finite subset A of X is independent and this can be done by induction on $n = |A|$. The case $n = 0$ is trivial. Assume $n = m + 1$ an let $A = \{a_{i_1}, \dots, a_{i_{m+1}}\}$ where $i_1 < \dots < i_{m+1} < \alpha$. By inductive hypothesis, $B = \{a_{i_1}, \dots, a_{i_m}\}$ is independent. Since $a_{i_{m+1}} \notin \mathrm{cl}(B)$, by *2* $A = B \cup \{a_{i_{m+1}}\}$ also is independent. □

Proposition 1.5 *Let (Ω, cl) be a pregeometry. Each of the following conditions is equivalent to $X \subseteq Z$ being a basis of Z:*

1. *X is independent and $\mathrm{cl}(X) = \mathrm{cl}(Z)$.*

2. *X is a minimal subset of Z such that $\mathrm{cl}(X) = \mathrm{cl}(Z)$.*

Proof: Let X be a basis of Z. Then $Z \subseteq \mathrm{cl}(X)$ and hence $\mathrm{cl}(Z) = \mathrm{cl}(X)$. Thus X satisfies the conditions of *1*. Such an X can not include a proper subset X' with $\mathrm{cl}(X') = \mathrm{cl}(Z)$ because any $a \in X \smallsetminus X'$ belongs to $\mathrm{cl}(X') \subseteq \mathrm{cl}(X \smallsetminus \{a\})$, contradicting the independency of X. Thus, *1* implies *2*. Assume now X is as in *2*. We will show it is a basis of Z. If $a \in X$ and $a \in \mathrm{cl}(X \smallsetminus \{a\})$ then $X \smallsetminus \{a\}$ is a proper subset of X with $\mathrm{cl}(X) = \mathrm{cl}(X \smallsetminus \{a\})$, a contradiction. Thus X is independent. If $a \in Z \smallsetminus X$ and $X \cup \{a\}$ is independent, then $a \notin \mathrm{cl}(X) = \mathrm{cl}(Z)$, which is impossible. Therefore X is maximally independent and X is a basis. □

Lemma 1.6 *Let (Ω, cl) be a pregeometry.*

1. *If X is minimal such that $a \in \mathrm{cl}(X)$, then for each $b \in X$, $(X \smallsetminus \{b\}) \cup \{a\}$ is a basis of $\mathrm{cl}(X)$.*

2. *If $X = Y \dot{\cup} Z$ is independent and Y' is a basis of $\mathrm{cl}(Y)$ then $Y' \cup Z$ is a basis of $\mathrm{cl}(X)$*

3. *If X is a basis of Z, $X' \subseteq X$ and $a \in Z \smallsetminus \mathrm{cl}(X')$, then there is some $b \in X \smallsetminus X'$ such that $(X \smallsetminus \{b\}) \cup \{a\}$ is a basis of Z.*

Proof: *1.* By minimality, X is independent. Also by minimality of X, $a \notin \mathrm{cl}(X \smallsetminus \{b\})$ and therefore $(X \smallsetminus \{b\}) \cup \{a\}$ is independent. By *P5* we know that $b \in \mathrm{cl}((X \smallsetminus \{b\}) \cup \{a\})$ and this implies $\mathrm{cl}((X \smallsetminus \{b\}) \cup \{a\}) = \mathrm{cl}(X)$.

2. For the independence of $Y' \cup Z$ apply point *3* of Lemma 1.4. On the other hand $\mathrm{cl}(X) = \mathrm{cl}(Y \cup Z) = \mathrm{cl}(\mathrm{cl}(Y) \cup Z) = \mathrm{cl}(\mathrm{cl}(Y') \cup Z) = \mathrm{cl}(Y' \cup Z)$.

We prove *3*. Clearly, $a \in \mathrm{cl}(X)$ and therefore there is some finite $X_0 \subseteq X$ of minimal size such that $a \in \mathrm{cl}(X_0)$. Since $a \notin \mathrm{cl}(X')$, $X_0 \nsubseteq X'$ and we can pick $b \in X_0 \smallsetminus X'$. By point *1* $(X_0 \smallsetminus \{b\}) \cup \{a\}$ is a basis of $\mathrm{cl}(X_0)$. By point *2* $((X_0 \smallsetminus \{b\}) \cup \{a\}) \cup (X \smallsetminus X_0)$ is a basis of $\mathrm{cl}(X) = \mathrm{cl}(Z)$. But $(X \smallsetminus \{b\}) \cup \{a\} = ((X_0 \smallsetminus \{b\}) \cup \{a\}) \cup (X \smallsetminus X_0)$. □

Proposition 1.7 *Let (Ω, cl) be a pregeometry.*

1. *Any independent subset X of Z can be extended to a basis X' of Z.*

2. *If X is a basis of Z and Y is an independent subset of Z, then there is some $X' \subseteq X$ such that $|X'| = |Y|$ and $(X \setminus X') \cup Y$ is a basis of Z.*

3. *If X, Y are bases of Z, then $|X| = |Y|$.*

Proof: *1* can be obtained as an application of Zorn's Lemma using only properties *P4* and *P2*. *2* can be easily proven using point *3* from Lemma 1.6. *3* is a direct consequence of point *2*. □

Definition 1.8 *If (Ω, cl) is a pregeometry and $A \subseteq \Omega$, we define the* localization of cl at A, cl_A, *by*

$$\mathrm{cl}_A(X) = \mathrm{cl}(A \cup X)$$

It is easy to check that (Ω, cl_A) is also a pregeometry. Let $X \subseteq \Omega$. We say that X is independent over A *(in (Ω, cl)) if X is independent in (Ω, cl_A).* A basis of X over A *(in (Ω, cl)) is a basis of X in (Ω, cl_A). We use the notation $\dim(X/A)$ for the dimension of $X \subseteq \Omega$ in (Ω, cl_A).*

Remark 1.9 *Let (Ω, cl) be a pregeometry and $X \subseteq \Omega$.*

1. *Any basis of X is also a basis of $\mathrm{cl}(X)$ and therefore*

$$\dim(X) = \dim(\mathrm{cl}(X))$$

2. *Since localizing at A or at $\mathrm{cl}(A)$ gives the same pregeometry,*

$$\dim(X/A) = \dim(X/\mathrm{cl}(A))$$

3. *For any $A \subseteq \Omega$, any basis of A can be completed with any basis of X over A to obtain a basis of $X \cup A$, and therefore*

$$\dim(X \cup A) = \dim(X/A) + \dim(A)$$

4. *If $Y \subseteq \Omega$ then any basis Z of $X \cap Y$ can be completed to a basis Z_1 of X and also to a basis Z_2 of Y and therefore*

$$\dim(X \cup Y) + \dim(X \cap Y) \leq \dim(X) + \dim(Y).$$

Moreover equality holds if $Z_1 \setminus Z$ is independent over Z_2.

Proof: In the case of *4*, note that

$$\begin{aligned} \dim(X) + \dim(Y) &= |Z_1| + |Z_2| \\ &= |Z| + (|Z_1 \setminus Z| + |Z_2|) \\ &= \dim(X \cap Y) + (|Z_1 \setminus Z| + |Z_2|) \end{aligned}$$

Now $X \cup Y \subseteq \mathrm{cl}((Z_1 \setminus Z) \cup Z_2)$, and hence $\dim(X \cup Y) \leq (|Z_1 \setminus Z| + |Z_2|)$. If $Z_1 \setminus Z$ is independent over Z_2 then $(Z_1 \setminus Z) \cup Z_2$ is a basis of $X \cup Y$ and since $(Z_1 \setminus Z) \cap Z_2 = \emptyset$ we get

$$\dim(X \cup Y) = |(Z_1 \setminus Z) \cup Z_2| = |Z_1 \setminus Z| + |Z_2|.$$

□

Proposition 1.10 *Let* (Ω, cl) *be a pregeometry and* $A, B \subseteq \Omega$*. If* A *is finite, then there exist some finite* $B_0 \subseteq B$ *such that* $\dim(A/B) = \dim(A/B_0)$

Proof: Let $X \subseteq A$ be a basis of A over B. Clearly X is independent over B' for all $B' \subseteq B$. For each $a \in A$, since $a \in \mathrm{cl}(XB)$ there is a finite subset $B_a \subseteq B$ such that $a \in \mathrm{cl}(XB_a)$. Let $B_0 = \bigcup_{a \in A} B_a$. For each $a \in A$, $a \in \mathrm{cl}(XB_0)$ and hence X is a basis of A over B_0. We conclude then that $\dim(A/B_0) = |X| = \dim(A/B)$. □

Definition 1.11 *A* geometry (Ω, cl) *is a pregeometry such that*

1. $\mathrm{cl}(\emptyset) = \emptyset$.
2. $\mathrm{cl}(\{a\}) = \{a\}$ *for every* $a \in \Omega$.

Definition 1.12 *If* (Ω, cl) *is a pregeometry, we define in* $\Omega' = \Omega \smallsetminus \mathrm{cl}(\emptyset)$ *an equivalence relation*

$$a \sim b \textit{ if and only if } \mathrm{cl}(a) = \mathrm{cl}(b).$$

Note that by the exchange property, for $a, b \in \Omega'$,

$$a \in \mathrm{cl}(b) \Leftrightarrow b \in \mathrm{cl}(a) \Leftrightarrow \mathrm{cl}(a) = \mathrm{cl}(b)$$

and therefore $\mathrm{cl}(a) \smallsetminus \mathrm{cl}(\emptyset) = a/\sim$*. Let us define for* $X \subset \Omega'/\sim$

$$\mathrm{cl}'(X) = \{a/\sim \, : a \in \mathrm{cl}(\bigcup X) \cap \Omega'\}$$

Then $(\Omega'/\sim, \mathrm{cl}')$ *is a geometry and it will be called the* canonical geometry *associated to the pregeometry* (Ω, cl).

Definition 1.13 *An* isomorphism *between the pregeometries* $(\Omega_1, \mathrm{cl}_1)$ *and* $(\Omega_2, \mathrm{cl}_2)$ *is a bijection* f *from* Ω_1 *onto* Ω_2 *which respects the closure operators:* $\mathrm{cl}_2(f(X)) = f(\mathrm{cl}_1(X))$ *for all* $X \subseteq \Omega_1$*. In other words:* X *is closed in* $(\Omega_1, \mathrm{cl}_1)$ *if and only if* $f(X)$ *is closed in* $(\Omega_2, \mathrm{cl}_2)$*. If they are the same pregeometry we talk of an* automorphism *of the pregeometry. A pregeometry* (Ω, cl) *is* homogeneous *if for each closed* $X \subseteq \Omega$*, for each* $a, b \in \Omega \smallsetminus X$ *there is an automorphism of* (Ω, cl) *which fixes pointwise* X *and sends* a *to* b.

Remark 1.14

1. *If a pregeometry* (Ω, cl) *is homogeneous, its localization* (Ω, cl_A) *at* $A \subseteq \Omega$ *is homogeneous.*
2. *If a pregeometry* (Ω, cl) *is homogeneous, its associated canonical geometry* $(\Omega'/\sim, \mathrm{cl}')$ *is homogeneous.*

Proof: It is an easy verification. □

Definition 1.15 *Let* (Ω, cl) *be a pregeometry. For* $A, B, C \subseteq \Omega$*, we say that* A is independent from C over B *if for all finite* $A_0 \subseteq A$, $\dim(A_0/BC) = \dim(A_0/B)$*. We write* $A \mathop{\smile\!\!\!\!\!|}^{\mathrm{cl}}_B C$ *for this.*

Remark 1.16 *Let* (Ω, cl) *be a pregeometry and let* $A, B, C \subseteq \Omega$*. Then* $A \mathop{\smile\!\!\!\!\!|}^{\mathrm{cl}}_B C$ *if and only if every* $X \subseteq A$ *independent over* B *is also independent over* BC.

Proposition 1.17 *Let* (Ω, cl) *be a pregeometry and let* $A, B, C \subseteq \Omega$*. Then*

1. $A \mathop{\smile\!\!\!\!\!\!\downarrow}_B^{\mathrm{cl}} \mathrm{cl}(B)$

2. *Normality: If* $A \mathop{\smile\!\!\!\!\!\!\downarrow}_B^{\mathrm{cl}} C$ *then* $A \mathop{\smile\!\!\!\!\!\!\downarrow}_B^{\mathrm{cl}} BC$.

3. *Base monotonicity: If* $A \mathop{\smile\!\!\!\!\!\!\downarrow}_B^{\mathrm{cl}} C$ *then* $A \mathop{\smile\!\!\!\!\!\!\downarrow}_{BD}^{\mathrm{cl}} C$ *for all* $D \subseteq C$.

4. *Monotonicity: If* $A \mathop{\smile\!\!\!\!\!\!\downarrow}_B^{\mathrm{cl}} C$ *then* $A' \mathop{\smile\!\!\!\!\!\!\downarrow}_B^{\mathrm{cl}} C'$ *for all* $A' \subseteq A$ *and all* $C' \subseteq C$.

5. *Finite character: If* $A_0 \mathop{\smile\!\!\!\!\!\!\downarrow}_B^{\mathrm{cl}} C$ *for all finite* $A_0 \subseteq A$, *then* $A \mathop{\smile\!\!\!\!\!\!\downarrow}_B^{\mathrm{cl}} C$.

6. *Transitivity: If* $A \mathop{\smile\!\!\!\!\!\!\downarrow}_B^{\mathrm{cl}} C$ *and* $A \mathop{\smile\!\!\!\!\!\!\downarrow}_{BC}^{\mathrm{cl}} D$, *then* $A \mathop{\smile\!\!\!\!\!\!\downarrow}_B^{\mathrm{cl}} CD$.

7. *Symmetry: If* $A \mathop{\smile\!\!\!\!\!\!\downarrow}_B^{\mathrm{cl}} C$, *then* $C \mathop{\smile\!\!\!\!\!\!\downarrow}_B^{\mathrm{cl}} A$.

8. *Local character: If* A *is finite, then for each* C *there is a finite* $B \subseteq C$ *such that* $A \mathop{\smile\!\!\!\!\!\!\downarrow}_B^{\mathrm{cl}} C$.

9. *Anti-reflexivity: If* $A \mathop{\smile\!\!\!\!\!\!\downarrow}_B^{\mathrm{cl}} A$, *then* $A \subseteq \mathrm{cl}(B)$.

Proof: *1* to *4* are straightforward. *5* and *8* follow from Proposition 1.10. *6* and *9* are also clear.

7. Note that the righthand version of finite character also holds: if $A \mathop{\smile\!\!\!\!\!\!\downarrow}_B^{\mathrm{cl}} C_0$ for all finite $C_0 \subseteq C$, then $A \mathop{\smile\!\!\!\!\!\!\downarrow}_B^{\mathrm{cl}} C$. By this and *4* we may assume that A, C are finite. Working over B, we may assume that $B = \emptyset$. Now, if $A \mathop{\smile\!\!\!\!\!\!\downarrow}^{\mathrm{cl}} C$, then $\dim(A) = \dim(A/C)$. Then

$$\dim(C/A) = \dim(CA) - \dim(A) = (\dim(C) + \dim(A/C)) - \dim(A) = \dim(C)$$

and hence $C \mathop{\smile\!\!\!\!\!\!\downarrow}^{\mathrm{cl}} A$. □

2 Modularity

Definition 2.1 *Let* (Ω, cl) *be a pregeometry.*

1. (Ω, cl) *is* trivial *or* degenerate *or* disintegrated *if* $\mathrm{cl}(X) = \bigcup_{a \in X} \mathrm{cl}(a)$ *for all nonempty* $X \subseteq \Omega$.

2. (Ω, cl) *is* modular *if the* modularity law

$$\dim(X) + \dim(Y) = \dim(X \cup Y) + \dim(X \cap Y)$$

holds for any closed sets X, Y.

3. (Ω, cl) *is* locally modular *if for some* $a \in \Omega$ *the localization* (Ω, cl_a) *is modular.*

4. (Ω, cl) *is* projective *if it is nontrivial and modular.*

5. (Ω, cl) *is* locally projective *if for some* $a \in \Omega$ *the localization* (Ω, cl_a) *is projective.*

6. (Ω, cl) *is* locally finite *if for any finite* $X \subseteq \Omega$, $\mathrm{cl}(X)$ *is finite.*

Proposition 2.2 *Each one of the defined properties of pregeometries (triviality, modularity,...) is possessed by* (Ω, cl) *if and only if it is possessed by its associated canonical geometry.*

Remark 2.3 *Trivial pregeometries are modular.*

Proposition 2.4 (Ω, cl) *is modular if and only if for all closed* X, Y*:* $X \mathop{\downarrow}^{\mathrm{cl}}_{X \cap Y} Y$

Proof: As in point *4* of Remark 1.9 fix Z a basis of $X \cap Y$, and $Z_1 \supseteq Z$ a basis of X, and $Z_2 \supseteq Z$ a basis of Y. If $X \mathop{\downarrow}^{\mathrm{cl}}_{X \cap Y} Y$ then $Z_1 \setminus Z$ (which is independent over Z) is independent over Z_2 and hence

$$\dim(X \cup Y) = |Z_1 \setminus Z| + |Z_2|$$

and

$$\dim(X) + \dim(Y) = \dim(X \cap Y) + \dim(X \cup Y).$$

Thus modularity is implied by this new condition. For the other direction we can clearly assume X, Y have finite dimension. In this case the modularity law implies

$$\dim(X \cup Y) = \dim(X) - \dim(X \cap Y) + \dim(Y) = \dim(X/X \cap Y) + \dim(Y).$$

Since also $\dim(X \cup Y) = \dim(X/Y) + \dim(Y)$ we conclude $\dim(X/Y) = \dim(X/X \cap Y)$ and hence $X \mathop{\downarrow}^{\mathrm{cl}}_{X \cap Y} Y$. □

Corollary 2.5 (Ω, cl) *is modular if and only if for all closed finite dimensional* X, Y*:*

$$\dim(X \cup Y) = \dim(X) + \dim(Y) - \dim(X \cap Y).$$

Proposition 2.6 (Ω, cl) *is modular if and only if for all* X, Y, A *such that* $A \subseteq X \cap Y$*:*

$$X \mathop{\downarrow}^{\mathrm{cl}}_{A} Y \text{ if and only if } \mathrm{cl}(X) \cap \mathrm{cl}(Y) \subseteq \mathrm{cl}(A).$$

Proof: Each implication will be based on Proposition 2.4. We first prove that this new condition implies modularity. Let X, Y be closed and let $A = X \cap Y$. Then $X \mathop{\downarrow}^{\mathrm{cl}}_{A} Y$.

For the other direction, it is always the case that $X \mathop{\downarrow}^{\mathrm{cl}}_{A} Y$ implies $\mathrm{cl}(X) \cap \mathrm{cl}(Y) \subseteq \mathrm{cl}(A)$. Now assume modularity and $\mathrm{cl}(X) \cap \mathrm{cl}(Y) \subseteq \mathrm{cl}(A)$. By modularity $X \mathop{\downarrow}^{\mathrm{cl}}_{\mathrm{cl}(X) \cap \mathrm{cl}(Y)} Y$ and since $\mathrm{cl}(X) \cap \mathrm{cl}(Y) \subseteq \mathrm{cl}(A) \subseteq \mathrm{cl}(Y)$, $X \mathop{\downarrow}^{\mathrm{cl}}_{\mathrm{cl}(A)} Y$. This clearly implies $X \mathop{\downarrow}^{\mathrm{cl}}_{A} Y$. □

Proposition 2.7 *For any pregeometry* (Ω, cl)*, those following are equivalent.*

1. (Ω, cl) *is modular.*
2. *If* $a \in \mathrm{cl}(Xc)$ *, then* $a \in \mathrm{cl}(bc)$ *for some* $b \in \mathrm{cl}(X)$*.*
3. *If* $a \in \mathrm{cl}(XY)$ *then* $a \in \mathrm{cl}(bc)$ *for some* $b \in \mathrm{cl}(X)$*,* $c \in \mathrm{cl}(Y)$*.*

Proof: *1* ⇒ *2*. Assume the pregeometry is modular. Let $a \in \mathrm{cl}(Xc)$. We may assume X has finite dimension. Choosing X minimal if necessary, we may assume $a, c \notin \mathrm{cl}(X)$. If $Y = \mathrm{cl}(ac)$, $\dim(Y/X) = 1$. If $a \in \mathrm{cl}(c)$ we are done. In other case $\dim(ac) = 2$. By modularity $\dim(\mathrm{cl}(X) \cap Y) = 1$. Hence there is some $b \in \mathrm{cl}(X) \cap Y \setminus \mathrm{cl}(\emptyset)$. Since $c \notin \mathrm{cl}(X)$, $c \notin \mathrm{cl}(b)$ and by exchange $b \notin \mathrm{cl}(c)$. Since $b \in \mathrm{cl}(ac) \setminus \mathrm{cl}(c)$, again by exchange, $a \in \mathrm{cl}(bc)$.

2 ⇒ *3*. We prove the claim for finite dimensional X, Y by induction on $\dim(XY)$. Let $a \in \mathrm{cl}(XY)$. By changing X, Y for minimal subsets if necessary, we may assume that for some closed $Z \subseteq Y$, for some $c \in Y$, $Y = \mathrm{cl}(Zc)$, $a \notin \mathrm{cl}(XZ)$ and $c \notin \mathrm{cl}(XZ)$. By *1* there is some $b \in \mathrm{cl}(XZ)$ such that $a \in \mathrm{cl}(bc)$. Since $XZ \subseteq XY$, $c \in Y$, and $c \notin \mathrm{cl}(XZ)$, we see that $\dim(XZ) < \dim(XY)$. By the inductive hypothesis there are $d \in X$, $e \in Z$ such that $b \in \mathrm{cl}(de)$. Then $a \in \mathrm{cl}(cde)$. Since $c, e \in Y$, by *2* $a \in \mathrm{cl}(df)$ for some $f \in Y$.

3 $\Rightarrow$ *1.* If modularity fails then there are closed finite dimensional X, Y such that $X \not\perp^{cl}_{X\cap Y} Y$. Then there is some $Y' \subseteq Y$ independent over $X \cap Y$ and not independent over X. For some $a \in Y'$, $a \in \mathrm{cl}((Y' \smallsetminus \{a\}) \cup X)$. If $Z = (Y' \smallsetminus \{a\}) \cup (X \cap Y)$, then $a \in \mathrm{cl}(Z \cup X)$ and $a \notin \mathrm{cl}(Z)$. By *3* $a \in \mathrm{cl}(bc)$ for some $b \in \mathrm{cl}(Z)$, $c \in X$. If $a \in \mathrm{cl}(b)$, then $a \in \mathrm{cl}(Z)$, which is not the case. Hence $a \in \mathrm{cl}(bc) \smallsetminus \mathrm{cl}(b)$ and by exchange $c \in \mathrm{cl}(ab) \subseteq Y$. Therefore $c \in X \cap Y \subseteq Z$, and then $a \in \mathrm{cl}(Z)$, a contradiction. □

Proposition 2.8 *If* (Ω, cl) *is an homogeneous pregeometry, the following are equivalent to the local modularity of* (Ω, cl)*:*

1. *For all closed* X, Y*, if* $\dim(X \cap Y) > 0$*, then* $\dim(X \cup Y) + \dim(X \cap Y) = \dim(X) + \dim(Y)$*.*
2. *For all finite dimensional closed sets* X, Y*, if* $\dim(X \cap Y) > 0$*, then* $\dim(X \cup Y) = \dim(X) + \dim(Y) - \dim(X \cap Y)$*.*
3. *For all closed* X, Y *if* $\dim(X \cap Y) > 0$*, then* $X \perp^{\mathrm{cl}}_{X\cap Y} Y$*.*
4. *For any* $a \in \Omega \smallsetminus \mathrm{cl}(\emptyset)$*, the localization* (Ω, cl_a) *is modular.*

Proof: The proof of Proposition 2.4 shows that in fact conditions *1*, *2* and *3* are equivalent.

Let $a \in \mathrm{cl}(\emptyset)$. Then every independent set X is also independent over a, $\mathrm{cl}_a(X) = \mathrm{cl}(X)$ and $\dim(X) = \dim(X/a)$.

Let $a \notin \mathrm{cl}(\emptyset)$. If $a \in \mathrm{cl}(X)$, then $\dim(X/a) + 1 = \dim(X)$. The reason is that we may find a basis Z of $\mathrm{cl}(X)$ with $a \in Z$ and then $Z \smallsetminus \{a\}$ is a basis of $\mathrm{cl}_a(X) = \mathrm{cl}(X)$ over a. In case $a \notin \mathrm{cl}(X)$ we have $\dim(X/a) = \dim(X)$. The reason now is that if Z is a basis of $\mathrm{cl}(X)$ then $Z \cup \{a\}$ is independent and hence Z is independent over a and therefore Z is also a basis of $\mathrm{cl}_a(X) = \mathrm{cl}(Xa)$ over a.

1 $\Rightarrow$ *4.* Let X, Y be two closed sets in (Ω, cl_a). Since $a \in X \cap Y$ and $a \notin \mathrm{cl}(\emptyset)$, $\dim(X \cap Y) > 0$. Then we can apply *1* to obtain $\dim(X \cup Y) + \dim(X \cap Y) = \dim(X) + \dim(Y)$. In this situation, dim coincides with $\dim(\ /a) + 1$ for all the sets involved and hence $\dim(X \cup Y/a) + \dim(X \cap Y/a) = \dim(X/a) + \dim(Y/a)$. This shows the modularity of (Ω, cl_a).

It is clear that *4* implies local modularity. Now we assume the localization (Ω, cl_a) is modular for some a and we prove *4*. If $a \in \mathrm{cl}(\emptyset)$, any closed set in (Ω, cl) is also closed in (Ω, cl_a) and the dimensions dim and $\dim(\ /a)$ coincide. Hence the geometry (Ω, cl) is modular in this case. Thus all its localizations are also modular. If $a \notin \mathrm{cl}(\emptyset)$, by homogeneity the localization at any other $b \notin \mathrm{cl}(\emptyset)$ is also modular.

4 $\Rightarrow$ *1.* Let X, Y be closed sets with $\dim(X \cap Y) > 0$. Then we can choose $a \in X \cap Y \smallsetminus \mathrm{cl}(\emptyset)$. Then X and Y are also closed in (Ω, cl_a) and $\dim(Z) = \dim(Z/a) + 1$ for $Z = X, Y, X \cap Y, X \cup Y$. By modularity of (Ω, cl_a), $\dim(X \cup Y/a) + \dim(X \cap Y/a) = \dim(X/a) + \dim(Y/a)$. It follows then that $\dim(X \cup Y) + \dim(X \cap Y) = \dim(X) + \dim(Y)$. □

3 Minimal types

Definition 3.1 *Let $A \subseteq M$ and let $\pi(x)$ be a type over A, where x is a single variable. We say that π is* minimal in M *if $\pi(M)$ is infinite and any relatively definable (in M) $X \subseteq \pi(M)$ is finite or cofinite. We say that $\pi(x)$ is* minimal *if it is minimal in the monster model $\mathfrak{C}$.*

Remark 3.2 *Let A, M, $\pi(x)$ be as above. Assume M is $|A|^+ + \omega$-saturated. Then π is minimal in M if and only if there is exactly one nonalgebraic extension $p(x) \in S(M)$ of π. Therefore, π is minimal in M if and only if it is minimal.*

Remark 3.3 *1. If $\varphi(x) \in L(A)$ axiomatizes the partial type $\pi(x)$ over A, then π is minimal if and only if φ is a strongly minimal formula.*

2. A complete type $p(x) \in S(A)$ is minimal if and only if $\mathrm{SU}(p) = 1$ and it is stationary.

Proof: For *2*, assume first p is minimal. Since p is not algebraic, $\mathrm{SU}(p) \geq 1$. Suppose $\mathrm{SU}(p) \geq 2$. Then p has a forking extension q over some set $B \supseteq A$ such that $\mathrm{SU}(q) \geq 1$, that is, q is nonalgebraic. Note that $\{\varphi(x) \in L(\mathfrak{C}) : |\varphi(\mathfrak{C}) \cap p(\mathfrak{C})| \geq \omega\}$ is a global extension of p that does not fork over A (for example because it does not split over A). Hence p has a nonforking extension q' over B. Again, it is nonalgebraic and hence we obtain two nonalgebraic extensions of p. Similarly for the stationarity of p since nonforking extensions of p are nonalgebraic.

Assume now p is a stationary type of SU-rank one. It is nonalgebraic. If p has two nonalgebraic extensions over $B \supseteq A$, by stationarity one of them is a forking extension and hence $\mathrm{SU}(p) \geq 2$, a contradiction. □

Remark 3.4 *For any sets A, B, the operator defined by $\mathrm{cl}(X) = \mathrm{acl}(XA) \cap B$ is a finitary closure operator on B.*

Definition 3.5 *Let $A \subseteq M$ and let $\pi(x)$ be a type over A, where x is a single variable. We say that $\pi(x)$ is* pregeometrical in M *if the closure operator defined in $\Omega = \pi(M)$ by $\mathrm{cl}(X) = \mathrm{acl}(XA) \cap \Omega$ for any $X \subseteq \Omega$ is a pregeometry. In the case $M = \mathfrak{C}$ we say that π is* pregeometrical.

Remark 3.6 *Let $\pi(x)$ be a partial type over A.*

1. *If π is pregeometrical, then it is pregeometrical in any model $M \supseteq A$.*
2. *If π is pregeometrical in some $|A|^+ + \omega$-saturated model $M \supseteq A$, then π is pregeometrical.*

Lemma 3.7 *Let $\pi(x)$ be a minimal type over A. For all $B \supseteq A$, if a, b, a', b' are realizations of π such that $a, a' \notin \mathrm{acl}(B)$, $b \notin \mathrm{acl}(Ba)$, and $b' \notin \mathrm{acl}(Ba')$, then $ab \equiv_B a'b'$.*

Proof: Note first that for any $B \supseteq A$ there is only one nonalgebraic type $p(x) \in S(B)$ extending π. Therefore, if $a, a' \notin \mathrm{acl}(B)$ realize π then $a \equiv_B a'$. Now let b, b' as indicated and choose b'' such that $ab \equiv_B a'b''$. Since $b', b'' \notin \mathrm{acl}(Ba')$ are realizations of π, they have the same type over Ba'. Hence $ab \equiv_B a'b'' \equiv_B a'b'$. □

Proposition 3.8 *Any minimal type is pregeometrical.*

Proof: Let π be over A. Assume the exchange property does not hold. Then for some $X \subseteq \pi(\mathfrak{C})$, there are realizations a, b of π such that $a \notin \mathrm{acl}(AX)$, $a \in \mathrm{acl}(AXb)$, and $b \notin \mathrm{acl}(AXa)$. Clearly, there is a sequence $(b_i : i < \omega)$ of different realizations $b_i \notin \mathrm{acl}(AX)$ of π for which there is some $c \models \pi$ such that $c \notin \mathrm{acl}(AX(b_i : i < \omega))$. Since $c \equiv_{AX} a$, there are also infinitely many realizations b' of π such that $b' \notin \mathrm{acl}(AX)$ and $a \notin \mathrm{acl}(AXb')$. By Lemma 3.7, all such b' have the same type over AXa and therefore are not algebraic over AXa. Again by Lemma 3.7 if b' is one of them, $ab \equiv_{AX} b'a$ and hence $a \notin \mathrm{acl}(AXb)$. $\square$

Lemma 3.9 *Let T simple and let $\pi(x)$ be a partial type over A in the single variable x. Assume for all $a \models \pi$, for all B, C such that $A \subseteq B \subseteq C$, $a \not\perp_B C$ iff $a \in \mathrm{acl}(C) \setminus \mathrm{acl}(B)$. Then π is pregeometrical.*

Proof: The exchange property follows from the hypothesis and the symmetry of forking independence in simple theories. $\square$

Definition 3.10 *A formula $\varphi(x)$ is* weakly minimal *if $D(\varphi(x)) = 1$. Recall that (i) $D(\varphi(x)) = 0$ iff $\varphi(x)$ is algebraic and (ii) if $\varphi(x) \in L(A)$ then $D(\varphi(x)) \geq \alpha + 1$ iff $D(\psi(x)) \geq \alpha$ for some $\psi(x) \vdash \varphi(x)$ which divides over A. D-rank coincides with Shelah continuous rank if T is stable.*

Remark 3.11 *1. Strongly minimal formulas are weakly minimal.*

2. Assuming T is stable, a nonalgebraic formula $\varphi(x) \in L(A)$ is weakly minimal if and only if for every $B \supseteq A$ there are at most $2^{|T|}$ nonalgebraic complete types $p(x) \in S(B)$ containing $\varphi(x)$.

Proof: *1.* A strongly minimal formula has Morley rank 1 and hence also D-rank 1.

2. In a stable theory every type has multiplicity bounded by $2^{|T|}$ and when a global type forks over a set A it has unboundedly many A-conjugates. $\square$

Proposition 3.12 *Let T be simple. Weakly minimal formulas and types of* SU*-rank* 1 *are pregeometrical.*

Proof: By Lemma 3.9. Let $A \subseteq B \subseteq C$. Note that in any case (i) if $a \not\perp_B C$, then $a \notin \mathrm{acl}(B)$ and (ii) if $a \in \mathrm{acl}(C) \setminus \mathrm{acl}(B)$, then $a \not\perp_B C$. Hence it only remains to show that (iii) if $a \not\perp_B C$, then $a \in \mathrm{acl}(C)$. This is clear in the case of SU-rank 1. Let us consider the case of a weakly minimal formula $\varphi(x) \in L(A)$. Assume $a \not\perp_B C$ where $\models \varphi(a)$ and $a \notin \mathrm{acl}(C)$. There is some $\psi(x) \in \mathrm{tp}(a/C)$ which forks over B. We can assume $\psi(x) \vdash \varphi(x)$. Since $a \notin \mathrm{acl}(C)$, $\psi(x)$ is not algebraic and hence $D(\psi(x)) > 0$. It follows that $D(\varphi(x)) > 1$. $\square$

Proposition 3.13 *Let $\pi(x)$ be a pregeometrical type over A and let us consider the pregeometry (Ω, cl), where $\Omega = \pi(\mathfrak{C})$ and $\mathrm{cl}(X) = \mathrm{acl}(XA) \cap \Omega$. Assume $\mathrm{SU}(a/A) \leq 1$ for all $a \in \Omega$.*

1. *$\dim(a_1, \ldots, a_n/B) = \mathrm{SU}(a_1, \ldots, a_n/AB)$ for any $B \subseteq \Omega$ and for any $a_1, \ldots, a_n \in \Omega$.*
2. *If T is simple, then for all subsets B, C, D of Ω, $B \perp_{AC} D$ iff $B \perp^{\mathrm{cl}}_C D$.*

Proof: *1.* By induction on n, using the assumption and Lascar inequalities.

2. For any $B, C, D \subseteq \Omega$,

$$
\begin{aligned}
B \mathop{\smile\!\!\!\!\!\!\,|}_{AC} D \quad & \text{iff} \quad \mathrm{SU}(\bar{b}/AC) = \mathrm{SU}(\bar{b}/ACD) \quad && \text{for all finite } \bar{b} \in B \\
& \text{iff} \quad \dim(\bar{b}/C) = \dim(\bar{b}/CD) \quad && \text{for all finite } \bar{b} \in B \\
& \text{iff} \quad B \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{cl}}_{C} D
\end{aligned}
$$

□

Remark 3.14 *1. The assumption of Proposition 3.13 is satisfied when π is minimal or it is a weakly minimal formula or it is a complete type of* SU*-rank* 1*.*

2. This section can be easily generalized to partial types $\pi(x)$ where x is an n-tuple of variables instead of being a single variable. The closure operator has to be defined as $\mathrm{cl}(X) = \mathrm{acl}(XA)^n \cap \Omega$*.*

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