On Normal Integer Parts of Real Closed Fields

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Introduction

- SHEPHERDSON [6]: The models of **Open Induction (OI)** are the **Integer Parts (IP)** of real closed fields (RCF).
- WILKIE [7]: Each discretely ordered $\mathbb{Z}$-ring can be embedded in a model of OI.
- VAN DEN DRIES [3]: The previous result is extended for the normal (integrally closed) case.
- MACINTYRE & MARKER [4]: Some classical theorems on primes fail in some nonstandard (normal) models of OI.
- MOURGUES & RESSAYRE [5]: Each RCF has an IP (the IP they constructed is not normal for non-trivial cases).
- S. KUHLMANN & AL. [2]: *Does every RCF have a normal IP?*
- BERARDUCCI and OTERO [1] constructed a nonstandard normal model of OI with cofinal set of primes. This model was an IP of $k(t)^{r.cl.}$ ($t \ll 1$, k is recursive RCF, $k \subseteq \mathbb{R}$, $\mathrm{trdeg}(k) = \aleph_0$).

Objectives

- to give a recurrent construction which allows to generate new IP's based on the existed ones.
- to construct normal IP's for a class of RCF's (a partial answer to a question in [2]).
- to show that each field from that class possesses an IP which satisfies the same homogeneous existential formulae (in the language without $<$) as a prescribed subfield $k_0 \subseteq \mathbb{R}$ with $\mathrm{trdeg}(\mathbb{R}/k_0) = 2^{\aleph_0}$.
- to present continuumly many elementary non-equivalent IP's for each field from the considered class.

Main Results

Proposition (Basic Construction). Let $K \subseteq F \subseteq_{\mathrm{tr}} K((G))$, M is an IP and H is a subfield of K and $M \subseteq H$. Also, let $\mu \stackrel{\mathrm{def}}{=} \mathrm{trdeg}(K/H) \geqslant |\mathrm{Neg}(F)|$ and $cf(\mu) > |\mathrm{Supp}(u)|$ for all $u \in \mathrm{Neg}(F)$. Then there exists $T \subseteq F \cap K((G^{\leqslant}))$ such that the elements of T are algebraically independent over H and $H[T]_0 \oplus M$ is an IP of F.

Theorem 1. Let G be a divisible ordered abelian (non-trivial) group with anti-well-ordered value set $\alpha^\star$. Let

- $\mathbb{R}(G) \subseteq F \subseteq_{\mathrm{tr}} \mathbb{R}((G))$ and
- $|F_\gamma| > |\gamma|$ $(\gamma \leqslant \alpha)$.

Then, assuming GCH, there exists an IP M of F such that $\mathrm{TH}_{\exists,h}(M) \equiv \mathrm{TH}_{\exists,h}(k_0)$.

Theorem 2. Under the same assumptions as in Theorem 1,

- The number of elementary non-equivalent IP's of F is continuum.
- F has a normal IP.

Theorem 3. Let F be an RCF with the residue field $\mathbb{R}$ and a value group G. Let G have an anti-well-ordered value set $\alpha^\star$ with $\alpha \leqslant \omega_1$. Then F has a normal IP, and the number of elementary non-equivalent IP's of F is continuum.

Definitions

- $\mathrm{Neg}(F) \stackrel{\mathrm{def}}{=} F \cap K((G^{<}))$, $G^{<} \stackrel{\mathrm{def}}{=} \{g \in G | g < 0\}$.
- $H[X]_0$ is the set of those polynomials of $H[X]$ whose constant term is 0, and $H[T]_0 = \{p(a_1,\ldots,a_n) \,|\, a_i \in T,\ p \in H[X]_0\}$.
- $\alpha^\star$ is an ordinal with its order reversed. We consider the groups G with value set $\alpha^\star$. We identify the natural valuation on G by the corresponding surjective map $v : G \to \alpha^\star$.
- $k_0 \subseteq \mathbb{R}$, $\mathrm{trdeg}(\mathbb{R}/k_0) = 2^{\aleph_0}$.
- Given $\mathbb{R}(G) \subseteq F \subseteq_{\mathrm{tr}} \mathbb{R}((G))$, we define $C_\gamma = \{g \in G \,|\, v(g) \leqslant \gamma\}$ $(\gamma \leqslant \alpha)$ and $F_\gamma = F \cap \mathbb{R}((C_\gamma))$.
- $\mathrm{TH}_{\exists,h}(M)$ denotes the part of existential theory of M (in the language without $<$) consisting of homogeneous formulae.

Remarks

- $\mathbb{R}((G))$ (with G as above) satisfies the hypotheses of Theorem 1.
- The field $k(t)^{r.cl.}$ has infinitely many elementary non-equivalent IP's.

Outline of the Proofs

1) Basic Construction. E is a transcendence base of K/H.

- choose a suitable well-order $\prec$ on E,
- induce functions $\|\cdot\| : K \to E$, $\|\cdot\| : \mathrm{Neg}(F) \to E$,
- construct a suitable well-order on $\mathrm{Neg}(F)$,
- define $\lambda : \mathrm{Neg}(F) \to E$ so that $\|u\| \prec \lambda(u)$ $(u \in \mathrm{Neg}(F))$.
- define $T_1 \subseteq \mathrm{Neg}(F)$ by transfinite induction.
- prove that $T \stackrel{\text{def}}{=} \{u + \lambda(u) | u \in T_1\}$ is the desired set.

2) The proof of Theorem 1 is by transfinite induction and is mainly based on Proposition. Set $k_0 = \mathbb{Q}(\{p^{1/p} | p \in A\})$ (A is a subset of primes) to prove Theorem 2 (for normality, set $A = \varnothing$). Theorem 3 is a direct consequence of Proposition.

Definitions

- K is an ordered field, G is an ordered abelian group (orders are total).
- A discretely ordered ring **(DOR)** is an ordered ring having 1 as its least positive element.
- A discretely ordered subring $M \subseteq K$ is called an **Integer Part (IP)** of K if for each $x \in K$ there exists $z \in M$ such that $z \leqslant x < z + 1$.
- M is called normal if $(x, y, c_1, \ldots, c_n \in M)$:

$$x^n + c_1 x^{n-1} y + \cdots + c_n y^n = 0 \Rightarrow \exists z \in M (x = yz).$$

- A subfield $F \subseteq K((G))$ is called truncation closed if: $\sum a_g t^g \in F$, $g_0 \in G \Rightarrow \sum_{g<g_0} a_g t^g \in F$. In symbols $F \subseteq_{\mathrm{tr}} K((G))$.
- **OI** is a first order theory in the language $\{0, 1, +, -, \cdot, <\}$ which has the axioms of **DOR** and the following scheme of axioms (ψ is quantifier-free):

$$\big(\psi(\vec{x}, 0) \wedge \forall y_{\geqslant 0} [\psi(\vec{x}, y) \to \psi(\vec{x}, y+1)]\big) \to \forall y_{\geqslant 0} \psi(\vec{x}, y)$$

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