CHARACTERIZING NIP AND NTP_2 SEMANTICALLY

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1. Basic definitions

Definition 1.1. 1) We say that $(\phi^i(x, y^i), \bar{a}^i : i < \kappa)$ with $|\bar{a}^i| = \omega$ is an independence pattern of depth κ if:
- for each $i < \kappa$:$\bigwedge_{j<\omega} \phi^i(x, a^i_j)$ is k^i-inconsistent for some $k^i < \omega$
- for each $f \in \omega^\kappa$: $\bigwedge_{i<\kappa} \phi^i(x, a^i_{f(i)})$ is consistent.
2) For theory T and $n < \omega$ define $\kappa^n_{inp}(T)$ to be the maximal possible depth of an independence pattern with $|x| \leq n$ or ∞ if it does not exist. Note that $\forall n < m$ $\kappa^n_{inp}(T) \leq \kappa^m_{inp}(T)$.
3) $\kappa_{inp}(T) := sup_{n<\omega}\{\kappa^n_{inp}(T)\}$
4) T is $strong^n$ if there is no independence pattern of infinite depth with $|x| \leq n$ and $strong$ if it is $strong^n$ for all n.
5) T has TP_2 (tree property of the second kind) if there is an independence pattern of infinite depth with some $\phi = \phi^i$ and $k = k^i$, and is NTP_2 if there is no such.

Note 1.2. 1) In the definition of independence pattern we can assume rows to be mutually indiscernible.
2) For any T: $\kappa_{inp}(T) \geq |T|^+ \iff \kappa_{inp}(T) = \infty \iff T$ has TP_2.

Remark 1.3. Strong theories were defined by Hans Adler in [Adler], all the rest is due to Shelah of course.

2. Bunch of acronyms

What is the place of all this in the complexity hierarchy of theories?

Fact 2.1. *1) simple* $\implies NTP_2$

2) $NIP \implies NTP_2$
3) strong $\implies NTP_2$ (and is actually a "uniform NTP_2", see further sections)
4) $NTP_2 + NSOP_2 = simple$
5) $strong + NIP = strongly\, dependent$
6) $strong + simple =$every finitary type has finite weight.

Proof. 1) and 2) is an easy exercise, 3) is a definition, 4) is in the book and 5), 6) are in [Adler].

□

3. Examples

Theories with NTP_2

(1) T_1, T_2 are $NTP_2 \implies T_1 \times T_2$ is NTP_2 (e.g. product of simple and dependent groups)
(2) Reduct of an NTP_2 theory is NTP_2
(3) So RV valued fields are NTP_2 assuming residue field and the value group are. Ask [MartinHils] to find out the details.
(4) Dividing-preserving expansion of NTP_2 theory is NTP_2 (we say that $T' \supseteq T$ is a dividing preserving expansion of T if $a \downarrow_c^d b$ in sense of $T \implies a \downarrow_c^d b$ in sense of T').
(5) 1-minimal expansions preserve NTP_2 (T' is a 1-minimal expansion of T if every formula from $L(T')$ in one variable is equivalent to some $L(T)$ formula, possibly with parameters. E.g. T' is strongly minimal iff it is a 1-minimal expansion of the infinite set and o-minimal if it is a 1-minimal expansion of DLO).

Reasonable examples with TP_2

(1) Triangle-free random graph
Well-known to be non-simple, but actually is even worse.
consider sequences $\{a_i^j b_i^j\}_{i<\omega}$ for $j < \omega$ such that $\forall i_1 \neq i_2\, a_{i_1}^j R b_{i_2}^j$

and these are the only edges around (easy to construct by genericity). Then it witnesses TP_2 for formula $xRy_1 \wedge xRy_2$ (where y_1 corresponds to a's and y_2 to b's).

(2) ω-free PAC fields
See [Chatzidakis] for the definitions and proof.

(3) Independent family of linear or circular orders
See [CheKap] for details.

(4) Any non-simple $NSOP_2$ theory.

4. Lifting mutual indiscernibility

The following observation on extracting mutually-indiscernible sequences is due to Itaï Ben Yaacov [BenYaacov].

Fact 4.1. *Let $C \subseteq \mathbb{M}$ and κ some cardinal. Then there is some λ such that for any $\bar{a}^{<n}$ with $|\bar{a}|^i \geq \lambda$ (and $|a^i_j| \leq \kappa$) we can find some mutually-indiscernible over C sequences $\bar{b}^{<n}$, $|b^i| = \omega$ such that $b^0_{<m}, ..., b^n_{<m} \equiv_C a^0_{\bar{j}_0}, ..., a^n_{\bar{j}_n}$ for some $\bar{j}_0 \in |\bar{a}_0|^m, ..., \bar{j}_n \in |\bar{a}_n|^m$.*

Note that for $n = 1$ this is normal "Erdös-Rado".
This and some more effort allows to show that

Theorem 4.2. *TFAE:*
1) $\kappa^n_{inp}(T) \leq \kappa$
2) $(*)^\kappa_n$: for any indiscernible array $\bar{a}^{<\kappa^+}$ and $c \in \mathbb{M}$, $|c| \leq n$ there is $h \in \kappa^+$ and an $a_0^{[h,\kappa^+)}$-automorphic image of $\bar{a}^{[h,\kappa^+)}$ indiscernible over c.

Now it is easy to see that $(*)^\kappa_1 \implies (*)^\kappa_2 \implies ... \implies (*)^\kappa_\kappa$, and so we can answer a question of Shelah from [ShC]:

Corollary 4.3. *$\kappa_{inp}(T) = \kappa^1_{inp}(T)$ (so in particular if a theory has TP_2 then there is some formula with a single variable witnessing it).*

Analogously we get

Corollary 4.4. *1) T is strong $\iff$ for any indiscernible array $\bar{a}^{<\omega}$ and finite c there is $h < \omega$ and an $a_0^{[h,\omega)}$-automorphic image of $\bar{a}^{[h,\omega)}$ indiscernible over c.*
2) Thus, strong=strong^1.

Problem 4.5. What strengthenings of this are true in simple or NIP theories?

5. Pseudo-local character

There are several equivalent pseudo-local characters characterizing NTP_2. It is still not clear which is the most useful.

Definition 5.1. We say that dividing in T has pseudo-local character with respect to a pre-independence relation $\downarrow$ when:
Let $p \in S(A), B \subseteq A$. Then there is some $C \subseteq A$, $|C| \leq |T|$ such that for each $D \subseteq A$: $D \downarrow_B BC \implies p|_{BD}$ does not divide over BC.

Note 5.2. Of course local character of dividing (so simplicity of a theory) implies pseudo-local character with respect to any pre-independence relation.

Example 5.3. If $M \models DLO$ and $p \in S(M)$, then p actually corresponds to a Dedekind cut of M. So though in general p is not definable, $p|_{\{c \in M: c \notin (a,b)\}}$ is definable whenever (a, b) is an interval of M which includes the cut. So p is definable in large pieces.

The following claim stated in a somewhat wrong way appeared first in [Sh783] and was proved there assuming NIP. Reading the proof shows that it actually follows from NTP_2, and is moreover equivalent to it.

Theorem 5.4. *TFAE: 1) T is NTP_2*

2) Dividing has pseudo-local character with respect to $\downarrow^{ist}$ *(invariant+non-coforking, for more on this see [CheKap])*
3) Dividing has pseudo-local character with respect to $\downarrow^{u}$ *for types over saturated enough models (where enough means* $p \in S(M)$ *and* M *is* $|B|^{+}$*-saturated)*
4) Dividing has pseudo-local character with respect to $\downarrow^{i}$*for types over saturated enough models*

5) If $(a_i : i < |T|^{+})$ *is a two ways-*$\downarrow^{u}$*-free sequence over* A *(or* M*) and* b *some tuple then there is some* $h < |T|^{+}$ *such that* $b \downarrow^{d}_{A} a_h$*.*

Problem 5.5. Does NTP_2 imply that dividing has pseudo-local character over arbitrary sets w.r.t. $\downarrow^{u}$?

Remark 5.6. 1) Recall that super-simple means that every type does not fork over some finite subset of its domain. Analogously strongness is characterized by being able to find not just small but actually a finite subset C as above.
2) In NIP we have in addition that dividing has psedo-local character with respect to $\downarrow^{d}$ over saturated enough models.

It also allows to characterize NTP_2 by counting small partial contradictory types with some extra technical conditions (which I hope to get rid of soon).

6. Abstract pre-independence relations

There is a well-known and useful characterization of simple theories due to Kim and Pillay as exactly those which admit an independence relation satisfying amalgamation of types. More precisely:

Theorem 6.1. *T is simple $\iff$ there is an invariant relation* $\downarrow$ *on the monster satisfying*
- right extension
- local character
- amalgamation over model (actually what one needs from it is that any pre-independence relation satisfying it implies non-dividing)

and $\mathop{\smile\hspace{-0.9em}\mid}$ *is exactly non-dividing/non-forking.*

There are several variants to give analogous characterizations for NTP_2 and NIP. Definitely none of them is final, but this is what I know is true at the moment

Theorem 6.2. *1) T is NIP $\iff$ there are two invariant relations $\downarrow$ and $\downarrow'$satisfying following*
- $\downarrow'$ exists over models and has extension on the right
- If $(a_i : i < |T|^+)$ is an $\downarrow'$-free sequence over A and b arbitrary tuple, then for some $h < |T|^+$ $b \downarrow_A a_h$ (or any pseudo-local characters)
- $\downarrow$ is bounded by the exponent, that is each type over model M has at most 2^M $\downarrow$-free extensions
2) If in addition $\downarrow$ it satisfies
- generic amalgamation of types: assume $p_i(x, a_i)$ is $\downarrow$-free over A for all $i < (2^{|T|+|A|})^+$. Then for some i, j we have that $p_i(x, a_i) \cup p_j(x, a_j)$ is $\downarrow$-free over A.
then it is exactly non-dividing.
3) If we put ω instead of $|T|^+$ then we get strongly dependent.

Of course it can be stated in terms of just $\downarrow$ by letting $\downarrow'$ be a shortcut for strict $\downarrow$-freenes.

Remark 6.3. What I call generic amalgamation of types is actually what is known as a chain condition. It is true both in simple theories and in NIP. Of course it should be true in NTP_2 as well, but no proof is known so far.

7. References

- [Adler] Hans Adler, "Strong theories, burden, and weight".
- [BenYaacov] Itaï Ben Yaacov, "Simplicity in compact abstract theories"

- [Chatzidakis] Zoe Chatzidakis, "Independence and imaginaries in (unbounded) PAC fields"
- [CheKap] Artem Chernikov, Itay Kaplan "Forking and dividing in NTP_2 theories"
- [MartinHils] Martin Hils, Université Paris 7
- [ShC] Saharon Shelah, "Classification theory"
- [Sh783] Saharon Shelah, "Dependent first order theories, continued"