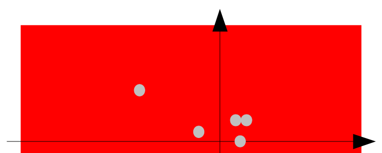
Definability in ample fields

Arno Fehm*
School of Mathematics, Tel Aviv University

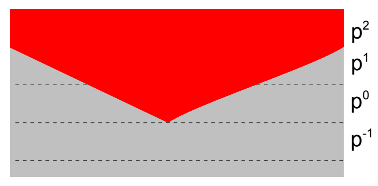
The **definable subsets** of 'big' fields, e.g. the classical model complete fields, tend to be large and well-behaved (unlike for example in $\mathbb{Q}$).
The following table shows some known results.

class of fields C	typical example $K \in C$	**definable subsets** $X \subset K$, $\|X\| = \infty$, $X \neq K$
ACF	$\mathbb{C}$	X is cofinite (Tarski)
RCF	$\mathbb{R}$	X is a finite union of intervals (Tarski)
p-adically closed	$\mathbb{Q}_p$	X contains an open subset (Macintyre)
pseudo-finite	$\prod \mathbb{F}_p /$ ultrafilter	X is not a field ([CvdDM92])
Henselian of char 0	$\mathbb{Q}((t))$	X is not a field ([JK08])



The definable subsets of an algebraically closed field are finite or cofinite.

In a real closed field, definable subsets are finite unions of intervals.



An infinite definable subset of a *p*-adically closed field has non-empty interior.

Look at the table and note two points:

Observation 1: In all of these cases, K has **no definable proper infinite subfields**.

Observation 2: In all of these cases, K is **ample**, i.e.
K is existentially closed in the field of formal power series $K((t))$.

This leads to the following question:

Ample (or large) fields were introduced in [Pop96] in the context of Galois theory.
They turned out to be useful in several other areas.
A lot of fields have already been proven to be ample, and many more are conjectured to be so.

A real field may have definable proper subfields.

Is it true that a perfect ample field has no definable proper infinite subfields?

No

as the following **counter-example** shows:

Let $\mathbb{Q}((X,Y))$ be the field of power series in two variables over $\mathbb{Q}$. Pop recently observed in [Pop08] that this field is ample. By results of Delon and others, $\mathbb{Q}$ is definable in $\mathbb{Q}((X,Y))$.

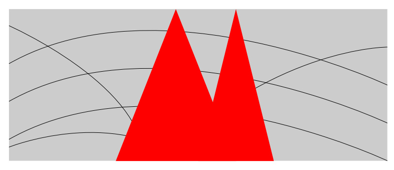
This example also gives a negative answer to the question in [JK08] if every perfect ample field is **very slim**.

But

a perfect ample field has **no <u>existentially</u> definable proper infinite subfields**.

More precisely, we show the following stronger result, which says that the existentially definable subsets are in some sense big with respect to subfields:

Theorem 1
Let K be a perfect ample field, and $X \subset K$ existentially K-definable and infinite.
Then for every proper subfield $L \subset K$, $|X \setminus L| = |K|$.



An infinite existentially definable subset of a perfect ample field is not contained in any proper subfield.

About the proof:

Theorem 1 follows from Theorem 2 below, which is purely algebraic. To understand the meaning of Theorem 2, note the following: A field K is ample if and only if every smooth K-curve C with $|C(K)| > 0$ satisfies $|C(K)| = \infty$. Pop has shown that in this case, even $|C(K)| = |K|$ holds, i.e. C has the maximal possible number of K-rational points. Theorem 2 strengthens this statement in two ways: First, this statement remains true if we subtract from $C(K)$ the set $C(L)$ of points that are rational over a given proper subfield $L \subset K$. Second, this statement remains true if we replace $C(K)$ by the image $f(C(K))$ of $C(K)$ under a certain given rational map f.

Theorem 2
Let K be an ample field, C a smooth K-curve with $|C(K)|>0$ and
$f: C \to D$ a separable non-constant K-rational map to an affine K-curve D.
Then for every proper subfield L of K, $|f(C(K)) \setminus D(L)| = |K|$.

Theorem 2 itself is proven by combining a trick from [Koe02] with some linear algebra.

References: [CvdDM92] Z. Chatzidakis, L. van den Dries, A. Macintyre, *Definable sets over finite fields*, Crelle, 1992.
[JK08] M. Junker, J. Koenigsmann, *Slim fields*, preprint, 2008.
[Koe02] J. Koenigsmann, *Defining transcendentals in function fields*, JSL, 2002.
[Pop96] F. Pop, *Embedding problems over large fields*, Ann. of Math., 1996.
[Pop08] F. Pop, *Henselian implies large*, preprint, 2008.

*supported by the European Commission under contract MRTN-CT-2006-035495
author's email: afehm@post.tau.ac.il