Second cohomology groups and finite covers

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Abstract

In this poster we show several applications of results of group cohomology to finite covers. In the first part we give a criterion for determining if a given profinite G-module is the kernel for a finite cover of some first order structure with automorphism group G. In the second part we provide several computations of certain cohomology groups and then, as an example of the effectivity of the criterion, we describe precisely all the covering expansions of certain free finite covers.

Preliminaries

If C is any set, then the full symmetric group Sym(C) on C can be considered as a topological group by giving it the topology whose open sets are arbitrary unions of cosets of pointwise stabilizers of finite subsets of C. Closed subgroups in this topology are precisely automorphism groups of relational structures on C.

Let C and W be two first-order structures. A finite-to-one surjection $\pi : C \to W$ is a **finite cover** if its fibres form an Aut(C)-invariant partition of C and the induced map $\mu : \mathrm{Aut(C)} \to \mathrm{Sym(W)}$ has image Aut(W). The *kernel* of π, K, is kerμ, which turns out to be a profinite group. The map μ is continuous and open and, thus,

$$1 \to K \to \mathrm{Aut(C)} \to \mathrm{Aut(W)} \to 1 \qquad (1)$$

is an exact sequence of topological groups.

If K is abelian, conjugation in Aut(C) gives an action of Aut(W) on K which makes K into a topological Aut(W)-module.

If C and C' are two structures with the same domain and $\pi : C \to W$ and $\pi' : C' \to W$ are finite covers with $\pi(c) = \pi'(c)$ for all $c \in C = C'$, then we say that π' is a **covering expansion** of π if $\mathrm{Aut(C')} \leq \mathrm{Aut(C)}$. The finite cover $\pi : C \to W$ is *minimal* if for all proper closed subgroups H of Aut(C), we have that $\mu(H) < \mathrm{Aut(W)}$.

Fact: Any finite cover $\pi : C \to W$ with kernel K has a covering expansion $\pi' : C' \to W$ which is minimal. Thus we can factorize $\mathrm{Aut(C)} = \mathrm{KAut(C')}$ and, to some extent, the natural problem of describing all the finite covers of a given structure reduces to finding the possible kernels and the minimal covers.

The lifting problem: Given $\pi_0 : C_0 \to W$ a finite cover with kernel K_0 and K a closed subgroup of K_0, decide if K is the kernel of any covering expansion of π_0.

In [1] the case when π_0 is a **principal** cover with abelian kernel K_0 was studied (in this case, $\mathrm{Aut(C_0)}$ splits over K_0,which makes the problem less hard).

Continuous cohomology groups

The appropriate cohomological machinary needed in order to work with finite covers is the continuous one (see [3]).

Let G be a permutation group considered with the canonical topology and K be a continuous G-module. We denote by $C_c^n(G, K)$ the additive group of continuous functions $\varphi : G^n \to K$. The usual coboundary operator δ_K^n sends $C_c^n(G, K)$ to $C_c^{n+1}(G, K)$, so that $(C_c^n(G, K); \delta^n)_{n \in \mathbb{N}}$ is a cochain complex. The cohomology of this complex, $H_c^*(G, K)$, is the *continuous cohomology* of G with coefficients in K.

Lemma 1 *If K is a **profinite** continuous G-module, cohomology of low degree retains its familiar applications: $H_c^1(G, K)$ classifies closed complements in the split extension and $H_c^2(G, K)$ classifies all permutation group extensions of K by G.*

First continuous cohomology groups have been previously used in the context of model theory. Indeed, several results in [1] and [5] show how $H_c^1(G, K)$ enable us to parametrise finite covers with kernel K of a fixed structure W with automorphism group G.

The main result

Theorem 2 *Let W be a first-order structure with automorphism group G. Suppose $\pi_0 : C_0 \to W$ is a finite cover with abelian kernel K_0 and K a closed G-submodule of K_0. Let*

$$0 \longrightarrow K \xrightarrow{\ i\ } K_0 \longrightarrow \bar{K} \longrightarrow 0$$

be the natural short exact sequence where i is the inclusion map. Consider

$$\cdots \longrightarrow H_c^1(G, \bar{K}) \longrightarrow H_c^2(G, K) \xrightarrow{\ i^*\ } H_c^2(G, K_0)$$

part of the long exact sequence, where i^* is the induced map in cohomology. Then there exists a covering expansion of π_0 with kernel K if and only if there exists an element $e \in H_c^2(G, K)$ such that $i^*(e) = e_0$, where e_0 is the element in $H_c^2(G, K_0)$ which gives rise, up to isomorphism, to $\mathrm{Aut(C_0)}$ *(as extension of K_0 by G).*

An application

Notation: $\mathbb{F}_p$: the integers modulo p; Ω: a countable set, $G = \mathrm{Sym}(\Omega)$; $[\Omega]^k$: the set of the k-subsets of distinct elements from Ω, where $k \in \mathbb{N}$.

Consider $[\Omega]^k$ as a structure with automorphism group G. Let $\pi_k : C_k \to [\Omega]^k$ be the free finite cover with binding groups $\mathbb{Z}_2$ and fibre groups $\mathbb{Z}_4$. It is easy to see that these free finite covers are non-splitting for every $k \in \mathbb{N}$.

Aim: Describe up to isomorphism all the covering expansions of π_k for $k = 2, 3$.

Fact: The kernels of the covering expansions of π_k are closed G-submodules of $\mathbb{F}_2^{[\Omega]^k}$.

The submodule structure of $\mathbb{F}_2^{[\Omega]^k}$

The natural action of G on $[\Omega]^k$ turns $\mathbb{F}_p[\Omega]^k$, the vector space over $\mathbb{F}_p$ with basis consisting of the elements of $[\Omega]^k$, into an $\mathbb{F}_pG$-module. The submodule structure of $\mathbb{F}[\Omega]^k$ is completely determined by the following maps.

Definition 3 *If $j \leq k$, there is a natural $\mathbb{F}_p Sym(\Omega)$-homomorphism*

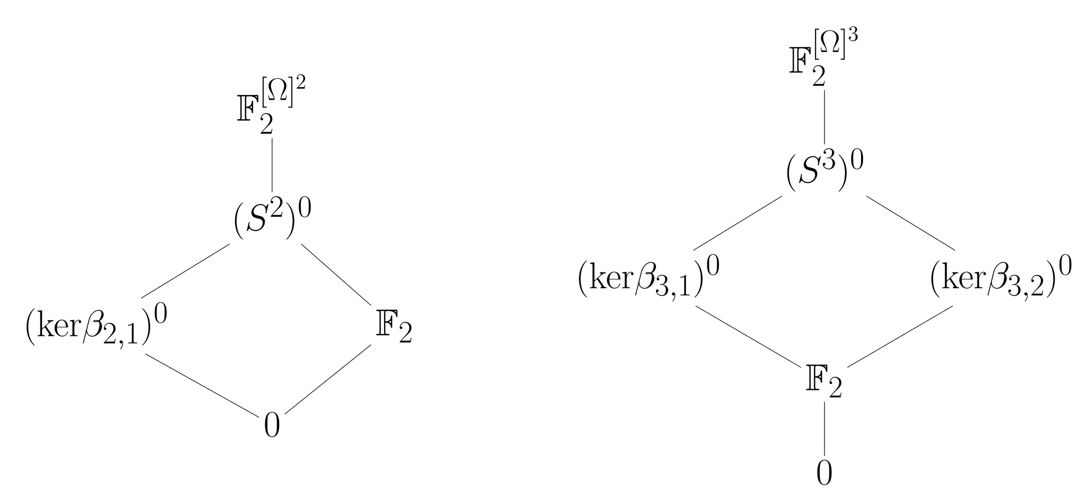
$$\beta_{k,j} : \mathbb{F}_p[\Omega]^k \to \mathbb{F}_p[\Omega]^j$$

given by $\beta_{k,j}(\omega) = \sum\{\omega' : \omega' \in [\omega]^j\}$ for $\omega \in [\Omega]^k$ and then extended linearly.

Every proper submodule of $\mathbb{F}_p[\Omega]^k$ is an intersection of kernels of the β-maps. In particular, the *Specht* submodule of $\mathbb{F}_p[\Omega]^k$, S^k, is caracterized as the intersection $\bigcap_{i=0}^{k-1} \mathrm{ker}\beta_{k,i}$. We denote by $\mathbb{F}^{[\Omega]^k}$ the dual module of $\mathbb{F}[\Omega]^k$ with the product topology.

The closed G-invariant subspaces of $\mathbb{F}^{[\Omega]^k}$ are precisely the annihilators X^0 in the Pontriagin duality of G-invariant subspaces X of $\mathbb{F}[\Omega]^k$ (see [4] and [6]).

The lattices of the closed submodules of $\mathbb{F}_2^{[\Omega]^k}$ for $k = 2, 3$ are the following.



Cohomological computations

We make use of the continuous versions of some results in the theory of cohomology of discrete groups, as the *Shapiro's Lemma* and long exact sequences (using the *dimension shifting* trick).

Lemma 4 *Let G be $Sym(\Omega)$ and $\mathbb{F}$ be a finite field. Then*

1. *$H_c^n(G, \mathbb{F}) = 0$ for every $n > 0 \in \mathbb{N}$ (see [3]).*
2. *$H_c^n(G, (\ker\beta_{2,1})^0) = H_c^n(G, (\ker\beta_{3,1})^0) = H_c^n(G, (S^2)^0) = 0$ for every $n > 0 \in \mathbb{N}$.*
3. *$H_c^n(G, (\ker\beta_{3,2})^0) = H_c^n(G, (S^3)^0) = \mathbb{F}_2$ for every $n > 0 \in \mathbb{N}$.*

Lemma 5 *Suppose G is a closed permutation group on Δ with a smooth strong type p. Suppose Q is a finite group and F a finite abelian group (regarded as a trivial G-module and Q-module). Then, for $n \geq 0$, $H_c^n(G \times Q, F) = H_c^n(Q, F)$.*

Corollary 1 *Let $G = Sym(\Omega)$. Then, for every $k \in \mathbb{N}$, $H_c^2(G, \mathbb{F}_p^{[\Omega]^k}) = H_c^2(Sym_k, \mathbb{F}_p)$. In particular, $H^2(G, \mathbb{F}_2^{[\Omega]^2}) = H^2(G, \mathbb{F}_2^{[\Omega]^3}) = \mathbb{F}_2$*

Describing the covering expansions

Theorem 6 *The free finite cover $\pi_2 : C_2 \to [\Omega]^2$ with binding groups $\mathbb{Z}_2$ and fibre groups $\mathbb{Z}_4$ is minimal.*

Theorem 6 solves the Problem 8.8 posed in [2].

Theorem 7 *The free finite cover $\pi_3 : C_3 \to [\Omega]^3$ with binding groups $\mathbb{Z}_2$ and fibre groups $\mathbb{Z}_4$ has only two distinct proper covering expansions (up to isomorphism). The kernels of the expansions are $(ker(\beta_{3,2}))^0$ and $(S^3)^0$. Moreover the expansion with kernel $(ker(\beta_{3,2}))^0$ is the only minimal one.*

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