Mild Parameterization in O-Minimal Structures

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1 Introduction

This work stems from an ongoing investigation into the **distribution of rational points** lying on particular sets in $\mathbb{R}^n$. This is a question within transcendental number theory, but it appears that the model theory of **o-minimal structures** is a very suitable context in which to consider it.

The origins of this current research lie in work of Bombieri and Pila ([1]), which established various estimates on the number of integral points on planar curves. The main theorem of this paper was recast by Pila, in [5], as a bound on the number of rational points with less than a certain **height**:

Definition 1.1. For any $\frac{a}{b} \in \mathbb{Q}$, with $\gcd(a, b) = 1$ and $b > 0$, we define its **height**, $H\left(\frac{a}{b}\right)$, to be $\max\{|a|, b\}$. The height of a finite tuple of rationals $(a_1, \ldots, a_m)$ is defined to be $\max_{1 \le i \le m}\{H(a_i)\}$. For any set $X \subseteq \mathbb{R}^n$, let $X(\mathbb{Q}, T)$ denote $\{\bar{q} \in X \cap \mathbb{Q}^n \mid H(\bar{q}) \le T\}$. We can define a natural distribution function thus: $N(X, T) := |X(\mathbb{Q}, T)|$.

Theorem 1.2 ([5]; Theorem 9)**.** *Let $f : [0, 1] \longrightarrow \mathbb{R}$ be a transcendental, real analytic function with graph X. For every $\epsilon > 0$, there is a constant $c(X, \epsilon)$ with the property that $N(X, H) \le c(X, \epsilon)H^\epsilon$.*

Following the above result, work has centred on higher dimensional sets. In this situation, a problem arises in trying to find an analogous result. Semialgebraic sets (i.e., those definable in $\bar{\mathbb{R}}$) of positive dimension may contain more than $H^{\epsilon'}$ rational points of height at most H, for some $\epsilon' > 0$, and so no bound of the above form can be obtained for these sets. Moreover, subsets $X \subseteq \mathbb{R}^n$ may contain semialgebraic subsets of positive dimension, without X itself having to be semialgebraic.

However, it is still possible to deduce some interesting, corresponding results, by making the following definition:

Definition 1.3. The **algebraic part** of a set $X \subseteq \mathbb{R}^n$, denoted X^{alg}, is the union of all connected, semialgebraic subsets of X of positive dimension. The **transcendental part** of X is the complement, $X \setminus X^{\text{alg}}$.

With these definitions, the same bound as in the Theorem above was obtained in [6] for the transcendental part of any compact, subanalytic set of dimension 2. The next goal was to generalize this to all globally subanalytic sets. These are precisely those sets definable in the o-minimal structure $\mathbb{R}_{\text{an}}$. Therefore, to tackle this problem, Pila and Wilkie set about proving the same bound for all sets definable in any o-minimal expansion of $\bar{\mathbb{R}}$.

Theorem 1.4 ([8]; Theorem 1.8)**.** *Let $\mathcal{R}$ be an o-minimal expansion of $\bar{\mathbb{R}}$ and let $X \subseteq \mathbb{R}^n$ be a set definable in $\mathcal{R}$. For each $\epsilon > 0$, there is a constant $c(X, \epsilon) \in \mathbb{R}$ such that $N(X \setminus X^{alg}, H) \le c(X, \epsilon)H^\epsilon$.*

Remark. In fact, for each $\epsilon > 0$, it is possible to find a definable set $X_\epsilon \subseteq X^{\text{alg}}$ such that the above holds with X_ϵ in place of X^{alg}.

In order to prove this theorem, the main result of [8] was an o-minimal version of a **reparameterization theorem** after Gromov ([2]).

Definition 1.5. In a fixed, o-minimal structure $\mathcal{M}$ over a real closed field M, we say that a definable subset $X \subseteq M^n$ is **strongly bounded** if there is a fixed finite bound $c \in \mathbb{N}$ (where $\mathbb{Q}$ is identified with the prime subfield of M) for all the coordinates of all the elements in X. (In particular, a set $\{x\} \subseteq M$ is strongly bounded if x is finite.)

Definition 1.6. For a fixed, o-minimal structure $\mathcal{M}$ over a real closed field M and some $r \in \mathbb{N}$, let X be a definable set in $\mathcal{M}$ of dimension k. An **r-parameterization** of X, for $r \in \mathbb{N} \cup \{\infty\}$, is a finite collection $\mathcal{S}_r$ of definable maps $\phi : (0, 1)^k \longrightarrow X$, satisfying:

- each map $\phi \in \mathcal{S}_r$ is in $\mathcal{C}^{(r)}((0, 1)^k)$;
- for each $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^k$ with $|\alpha| \le r$ and for each map $\phi \in \mathcal{S}_r$, $|D^\alpha \phi| = \left|\frac{\partial^{|\alpha|}\phi}{\partial x_1^{\alpha_1} \ldots \partial x_m^{\alpha_m}}\right|$ is strongly bounded;
- $X = \bigcup_{\phi \in \mathcal{S}_r}(\phi((0, 1)^k))$.

A definable function $f : X \longrightarrow M^n$ is said to have an **r-reparameterization** if the graph of f has an r-parameterization.

Theorem 1.7 (**Reparameterization Theorem**; [8]; Theorems 2.3 & 2.5)**.** *Let $\mathcal{M}$ be a fixed, o-minimal structure over a real closed field M and let X be a strongly bounded set. For any $r \in \mathbb{N}$, X has an r-parameterization.*

If $f : X \longrightarrow M$ is a definable map such that both X and the range of f are strongly bounded, then, for any $r \in \mathbb{N}$, f has an r-reparameterization.

The reparamaterization theorem can be used to show that the collection of rational points of height $\le H$ lying on certain definable families of fibres is contained in $O(H^\epsilon)$ **hypersurfaces** of some degree d, i.e. in few sets of the form $\{\bar{x} \in \mathbb{R}^n \mid p(\bar{x}) = 0\}$, for $p : \mathbb{R}^n \longrightarrow \mathbb{R}$ a non-zero polynomial over $\mathbb{R}$ of degree d. From this it is possible to prove a version of Theorem 1.4 for fibres, by induction on the fibre dimension of a given family of fibres Z.

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2 Mild Parameterization

The relationship between parameterization and the number of hypersurfaces which contain the rational points of a given definable set is a significant one for us. Further conditions placed on the nature of a parameterization can, in some circumstances, lead to significant improvements in the bound on $N(X, H)$.

In particular, the notion of **mild functions** was introduced by Pila in [7] to approach the special case in which $X \subseteq \mathbb{R}^2$ is a **Pfaff curve**, which is the graph of a Pfaffian function on some connected subset of its domain. We give here a slightly more general form of the definition of mildness, which makes sense in any o-minimal structure $\mathcal{M}$ over a real closed field M and is appropriate to functions defined on any open domain.

Definition 2.1. We say that a smooth map $f : U \longrightarrow M$, on an open subset U of M^m, is **mild** if there exist $B, C \in M$ with $B, C > 0$ such that $|D^\alpha f(\bar{x})| \le a!(B|\alpha|^C)^{|\alpha|}$, for all $\alpha \in \mathbb{N}^m$ and for all $\bar{x} \in U$. In case of any ambiguity, we may sometimes say that f is (B, C)-mild.

A **mild parameterization** is an ∞-parameterization $\mathcal{S}_\infty$ for which all $\phi \in \mathcal{S}_\infty$ are mild.

The main result of [7] is then the following.

Theorem 2.2 ([7]; Theorem 1.5)**.** *Let $X \subseteq [-1, 1]^2 \subseteq \mathbb{R}^2$ be a Pfaff curve with a mild parameterization. There are constants $\beta, \gamma > 0$ such that, whenever $H \ge e$, $N(X, H) \le \beta(\log H)^\gamma$.*

This is a significant improvement on the bound on $N(X, H)$ and is due to having such bounds on all derivatives of the parameterizing functions. If these functions are all (B, C)-mild, this ensures that $X \cap \mathbb{Q}^2$ is contained in the union of a most $O((\log H)^{2C})$ algebraic curves of degree $d = [\log H]$ (the largest integer $\le \log H$).

In light of this result, the natural conjecture raised in [7] is then the following:

Conjecture ([7]; Conjecture 1.6)**.** *Every Pfaff curve $X \subseteq [-1, 1]^2 \subseteq \mathbb{R}^2$ has a mild parameterization.*

3 Locally Polynomially Bounded Structures

Our aim is to shed light on Pila's Conjecture 1.6 by working within the context of o-minimal expansions of $\bar{\mathbb{R}}$. Since the Reparameterization Theorem (1.7) gives us that all definable, strongly bounded sets in such structures have an r-parameterization, for any $r \in \mathbb{N}$, one approach might be to seek out examples of such structures in which all definable, strongly bounded functions f are at least piecewise mild. There is already some evidence for this being true in $\mathbb{R}_{\text{exp}}$: in [7], Pila showed that, for any $m \in \mathbb{N} \setminus \{0\}$, the map $\phi : (0, 1) \longrightarrow \mathbb{R}$; $\phi(x) = \exp\left(-\frac{1}{x^m}\right)$ is mild. Note that the structure $\mathbb{R}_{\text{exp}}$ is an example of a **Locally Polynomially Bounded (LPB)** structure:

Definition 3.1 ([3]; Section 3)**.** Fix $\mathcal{M} = \langle \bar{M}, \mathcal{F} \rangle$, an o-minimal, model complete structure expanding a real closed field $\bar{M}$, where $\mathcal{F}$ is a collection of total, smooth functions $f : M^n \longrightarrow M$ for various n. Define $\mathcal{F}_{\text{res}} := \{f\restriction_B \mid f \in \mathcal{F}, B \text{ an open box in dom}(f)\}$. We say that $\mathcal{M}$ is **Locally Polynomially Bounded** if $\langle \bar{M}; \mathcal{F}_{\text{res}} \rangle$ is polynomially bounded.

In these structures, the 0-definable functions have very nice representations in terms of the functions in $\mathcal{F}$, which gives us a lot of information to work with when considering this question of mildness of definable functions.

Definition 3.2. In an LPB structure $\langle \bar{M}, \mathcal{F} \rangle$, let $\tilde{\mathcal{F}}$ denote the smallest collection of functions, containing both $\mathcal{F}$ and all polynomials over $\mathbb{Q}$, which is closed under $\mathbb{Q}$-algebra operations and partial differentiation. For every $n \ge 1$, let R_n denote the $\mathbb{Q}$-algebra of n-ary functions in $\tilde{\mathcal{F}}$. A 0-definable function $f : U \longrightarrow M$ on an open set $U \subseteq M^n$, is **implicitly $\mathcal{F}$-defined** if there exist $m \ge 1$, functions $g_1, \ldots, g_m \in R_{n+m}$ and 0-definable maps $\phi_1, \ldots, \phi_m : U \longrightarrow M$ such that

- $f = \phi_i$ for some $i \in \{1, \ldots, m\}$;
- $\langle \phi_1(\bar{x}), \ldots, \phi_m(\bar{x}) \rangle$ is a regular zero of the system given by $g_1(\bar{x}, \cdot), \ldots g_m(\bar{x}, \cdot)$, for all $\bar{x} \in U$.

Theorem 3.3 ([3]; Corollary 4.5)**.** *Let $f : U \longrightarrow M$ be a 0-definable function on an open set $U \subseteq M^n$. There are 0-definable open sets $U_1, \ldots, U_k \subseteq U$ with $\dim(U \setminus \cup_{i=1}^k U_i) < n$ such that each map $f\restriction_{U_i}$, for $i \in \{1, \ldots, k\}$, is implicitly $\mathcal{F}$-defined.*

Given this representation of 0-definable functions, we consider whether, in the context of LPB structures, mildness is preserved under being implicitly $\mathcal{F}$-defined, were all functions in $\mathcal{F}$ mild. It is relatively straightforward to show that mildness is preserved under composition, partial differentiation and multiplication of functions, so this seems plausible.

However, we soon see that if $f : U \longrightarrow M$, for $U \subseteq M^n$ an open set, is implicity $\mathcal{F}$-defined from mild maps $g_1, \ldots, g_m \in R_{n+m}$, as part of a collection of 0-definable functions $\phi_1, \ldots, \phi_m : U \longrightarrow M$, then, for f to be mild, we need both that all $\phi_1, \ldots, \phi_m$ are bounded and that $\langle \phi_1(\bar{x}), \ldots, \phi_m(\bar{x}) \rangle$ is a regular zero of the system given by $g_1(\bar{x}, \cdot), \ldots g_m(\bar{x}, \cdot)$, **for all $\bar{x}$ in the closure of U**. However, this is not guaranteed; the Jacobian of $g_1(\bar{x}, \cdot), \ldots g_m(\bar{x}, \cdot)$ may tend to zero at the boundary of U.

4 Quasianalyticity and o-minimal structures

Therefore, we now seek to establish an explicit construction of an LPB structure (in fact, first, just a model complete, o-minimal, polynomially bounded structure) which does **not** have mild parameterization (i.e. defines a function without it). To do this we follow the methods of Rolin, Sanz and Schäfke, in [9], and of Le Gal and Rolin, in [4].

Let us explain how we may define any smooth function $h : [-1, 1] \longrightarrow \mathbb{R}$ in some model complete, o-minimal, polynomially bounded expansion of the real field $\bar{\mathbb{R}}$, as long as its Taylor series at the origin has coefficients algebraically independent over $\mathbb{Q}$ and there is a function $g : U \longrightarrow \mathbb{R}$, with U an open neighbourhood of $[-1, 1]$, which is analytic, except perhaps at the origin, and such that $h = g\restriction_{[-1,1]}$.

We start by constructing a **quasianalytic class $\mathcal{A}$ of smooth germs**, which we ensure is the smallest class of smooth germs which

- contains both the germ at the origin of h and, for every $n \in \mathbb{N}$, the germ at the origin in $\mathbb{R}^n$ of every function in n variables which is analytic at the origin;
- is closed under sums, products, monomial division, composition and defining functions implicitly.

It is possible to do this in such a way that its members are actually generated from the germ at the origin of h. To make this formal, we adopt the notion of **operators** - closure maps on germs - as described in [4]. It is the algebraic independence over $\mathbb{Q}$ of the Taylor coefficients at the origin which ensures that this is a quasianalytic class.

From this class $\mathcal{A}$ we may define the notion of $\mathcal{A}$-analytic in the following manner.

Definition 4.1. A function $F : U \longrightarrow \mathbb{R}$ on an open subset $U \subseteq \mathbb{R}^n$ is said to be **$\mathcal{A}$-analytic** if, for any $\bar{a} \in U$, there exists a germ $f_{\bar{a}} \in \mathcal{A}_n$ such that the germ of F at $\bar{a}$ is equal to the germ of the map $\bar{x} \mapsto f_{\bar{a}}(\bar{x} - \bar{a})$ at $\bar{a}$.

It is possible to show that all germs in our class $\mathcal{A}$ have an $\mathcal{A}$-analytic representative (by minimality of $\mathcal{A}$ and the fact that the germ at the origin of h has representative which is analytic in a neighbourhood of $[-1, 1]$, except perhaps at the origin). Using this class of representatives, we may define the notions of $\mathcal{A}$-semianalytic and globally $\mathcal{A}$-subanalytic sets in the obvious manner. These give us a **Theorem of the Complement**, i.e. that the complement of any globally $\mathcal{A}$-subanalytic set is globally $\mathcal{A}$-subanalytic.

We are now in a position to state an o-minimality and model completeness result. First define $\mathcal{H} := \{\tilde{h}^{(i)} \mid i \in \mathbb{N}\}$, where $\tilde{h}^{(i)}$ is given by

$$\tilde{h}^{(i)} = \begin{cases} h^{(i)}(\bar{x}) & \text{if } \bar{x} \in [-1, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.2. *The structure $\mathbb{R}_{an,\mathcal{H}} := \langle \mathbb{R}_{an}; \mathcal{H} \rangle$, is o-minimal and model complete.*

Note that if $\mathbb{R}_{\text{an},\mathcal{H}} = \langle \bar{\mathbb{R}}, \mathcal{G} \rangle$, and $\mathcal{F} := \{g(\tau_n(\bar{x})) \mid g \in \mathcal{G}\}$, where $\tau_n : (x_1, \ldots, x_n) \mapsto \left(\frac{x_1}{\sqrt{1+x_1^2}}, \ldots, \frac{x_n}{\sqrt{1+x_n^2}}\right)$, for appropriate $n \in \mathbb{N}$, then $\mathbb{R}_{\mathcal{F}} := \langle \bar{\mathbb{R}}, \mathcal{F} \rangle$ is an LPB structure in which h is definable.

Remark. Since the function $h \circ \tau_1$ is analytic, except perhaps at the origin, by using the piecewise implicit $\mathcal{F}$-definition of 0-definable functions in $\mathbb{R}_{\mathcal{F}}$, it is possible to show that $\mathbb{R}_{\mathcal{F}}$ has analytic cell decomposition.

5 Without Mild Parameterization

So now it only remains to find a function $H : \mathbb{R} \longrightarrow \mathbb{R}$ without mild parameterization which is also given by $x \mapsto h(\tau_1(x))$, for some $h : \mathbb{R} \longrightarrow \mathbb{R}$ as described in Section 4.

To this end we employ the **Whitney Extension Theorem**. This allows us to describe a function h, which is analytic, except perhaps at the origin, and has suitable derivatives, not only at the origin, but also at a sequence of points approaching the origin, to confound the bounds of mild parameterization with sufficiently wild behaviour. It turns out that it is enough to ensure that $H^{(n)}(\frac{1}{n}) = n^{n^5}$, but note that in order to prove that this function does not have mild parameterization, our proof makes key use of the polynomially boundedness of the structure $\mathbb{R}_{\text{an},\mathcal{H}}$.

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