Small groups of odd type

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A small group of finite Morley rank

PSL_2

A closer view

$$\mathrm{PSL}_2$$

In this section:

1. Groups and rank
   - Groups, rank, and algebraic groups
   - Groups of low Morley rank
   - Groups of finite MR and finite groups

2. PSL_2
   - Early results
   - Description
   - Analysis

3. Results
   - The notion of smallness and results
   - Difficulties and solutions
   - The main tool

$\aleph_1$-categorical groups

Groups of finite Morley rank appeared as $\aleph_1$-categorical groups.

Theorem (Baldwin, Zilber)

A simple group has finite Morley rank iff it is $\aleph_1$-categorical.

In the 80's, Borovik and Poizat suggested a more naive approach.

Theorem (Poizat)

A group has finite Morley rank iff there is a rank function rk on the set of interpretable sets, which behaves like a dimension ought to.

Morley rank and Zariski dimension

- Typical example of a group of finite Morley rank :

  an alg. group over an alg. closed field,
  equipped with the Zariski dimension.

- an infinite field of finite Morley rank is alg. closed (Macintyre)
- slogan :

  groups of finite Morley rank generalize
  alg. groups ranked by the Zariski dimension

Ranked groups and algebraic groups

- Analogies :
  - chain conditions
  - connected components for definable subgroups "H°"
  - generation lemmas (in part., G' is definable!)
  - presence of a field (sometimes)

Conjecture (Cherlin-Zilber)

A simple infinite group of finite Morley rank is (isomorphic to) an algebraic group over an algebraically closed field.

Rank 1 and 2

Let us attack the conjecture inductively.

Fact: There are no simple groups of Morley rank 1 or 2.

- Groups of Morley rank 1 are abelian (Reineke).
- Groups of Morley rank 2 are solvable (Cherlin).

Now what about groups of rank 3?

Rank 3 and PSL_2

- Some tapas:
  - $\mathrm{SL}_2 = \{M \in \mathrm{GL}_2 : \det M = 1\}$
  - $Z(\mathrm{SL}_2) = \{\pm \mathrm{Id}\}$
  - $\mathrm{PSL}_2 = \mathrm{SL}_2 / Z(\mathrm{SL}_2)$

  PSL_2 is the smallest simple algebraic group:
  Zariski dimension = 3, Lie rank = 1, Morley rank $= 3\,\mathrm{rk}\,\mathbb{K}$
- PSL_2: only simple algebraic group of Zariski dimension 3
- PSL_2: only simple algebraic group of Lie rank 1
- PSL_2 is the basis of inductive arguments $\rightarrow$ crucial piece

Main question of the talk:

Identify PSL_2 among small groups of finite Morley rank

Rank 3 and bad groups

Theorem (Cherlin)

A simple group of MR 3 is either $\mathrm{PSL}_2(\mathbb{K})$ or a simple bad group.

- A bad group would be a weird non-algebraic configuration.
  - No fields involved.
  - Disjoint union of maximal subgroups.
  - No involutions.
- Open for 30 years!
- Moral:

"low Morley rank" not a good notion of smallness

Groups of finite Morley rank and groups of Morley rank 0

Conjecture (Cherlin-Zilber)

A simple infinite group of finite Morley rank is an algebraic group over an ACF.

Theorem (A logician's CFSG)

A simple group of Morley rank 0 *is*

- *the finite version of an algebraic group*
- *or something else.*

Well... you know logicians.

Was the previous slide sabotage?

Theorem (CFSG)

A finite simple group is

- *cyclic* $\mathbb{Z}/p\mathbb{Z}$
- *alternate* A_n
- *the finite version of an alg. group (Chevalley twists welcome)*
- *or one of* 26 *"sporadic" known exceptions.*

- the only infinite cyclic group, $\mathbb{Z}$, is not ω-stable
- the infinite version of A_n is not stable (not M_C)
- fields of finite Morley rank do not allow Chevalley twists
- the sporadics *may* disappear when one goes to infinite objects

Borovik's program

- The Cherlin-Zilber Conjecture looks like a simpler CFSG

idea (Borovik): imitate CFSG

- (possible gain: a "generic", simpler CFSG)
- Work with 2-elements, involutions, and their centralizers
- fortunately: good 2-Sylow theory

Four types

- Let S be a Sylow 2-subgroup. Then $S^\circ = U * T$, with
  - U of bounded exponent is 2-unipotent
    i.e. definable, connected, of exponent 2^k
  - $T \simeq \mathbb{Z}_{2^\infty}^d$ is a 2-torus of Prüfer rank d
    $\mathbb{Z}_{2^\infty}$ is the Prüfer 2-group $\{z \in \mathbb{C} : z^{2^k} = 1 \text{ for some } k \in \mathbb{N}\}$
- One thus defines 4 "types" depending on structure of S°

	$T = 1$	$T \neq 1$
$U = 1$	$2^\perp$	odd
$U \neq 1$	even	mixt

- correspond to the char. of the expected underlying field

State of the Case-Division

- Cases $U \neq 1$ have been solved (Altınel, Borovik, Cherlin).
- Cases $U = 1$ are open.
- The case $U = T = 1$ looks so hard the Conjecture might fail.
  - no Feit-Thompson Theorem

    FT: finite simple groups have involutions... (would kill bad groups!)

Yet one can work in *odd type* $S^\circ \simeq \mathbb{Z}_{2^\infty}^d$ $(U = 1$ but $T \neq 1)$.

Problem: Identify PSL_2 among small groups of odd type.

In this section:

The Hrushovski analysis

Theorem (Hrushovski)

Let a non-solvable group of finite MR G act definably and faithfully on a strongly minimal set. Then $G \simeq \mathrm{PSL}_2$ and $\mathrm{rk}\, G = 3$.

In practice, actions arise from coset spaces.

Corollary (Cherlin)

Let G be a non-solvable group of finite Morley rank with a definable subgroup of corank 1. Then $G \simeq \mathrm{PSL}_2$ (and $\mathrm{rk}\, G = 3$).

Moral: try to understand the action on coset spaces

Delahan-Nesin identification

Caution: this slide contains technical material.

Another identification result using actions.

Theorem (Delahan-Nesin)

Let G be a group of finite Morley rank. Assume that G is an infinite split Zassenhaus group. Assume further that the stabilizer of two points contains an involution. Then $G \simeq \mathrm{PSL}_2$.

A Zassenhaus group is a 2-transitive group (G, X) s.t. $G_{x,y,z} = 1$.
It is split if there is $N \lhd G_x$ s.t. $G_x = N \rtimes G_{x,y}$.

The setting

- Moral of last slide: useful abstract identification results exist
- From now on it will suffice to
  - fix an involution $i \in G$
  - fix a Borel $B \geq C^\circ(i)$

    Recall that a Borel is a maximal definable, connected, solvable subgroup
  - split $B \simeq \mathbb{K}_+ \rtimes \mathbb{K}^\times$
  - understand G/B
- Nesin's machinery can then recognize PSL_2
  - Question: find natural properties of PSL_2 characterizing it
- Latin letters for the abstract group; Greek for the true PSL_2.

Study of PSL_2

Let $\mathbb{K} \models \mathrm{ACF}_{\neq 2}$. Let's have a look at $\mathrm{PSL}_2(\mathbb{K})$.

- $\iota = \begin{pmatrix} i & \\ & -i \end{pmatrix}$
- $\beta = \left\{ \begin{pmatrix} t & a \\ & t^{-1} \end{pmatrix}, a \in \mathbb{K}, t \in \mathbb{K}^\times \right\} > C^\circ(\iota)$ is a Borel
- $\beta' = F^\circ(\beta) = \left\{ \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix}, a \in \mathbb{K} \right\} \simeq \mathbb{K}_+$
- $\Theta = \left\{ \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}, t \in \mathbb{K}^\times \right\} \simeq \mathbb{K}^\times$
- Then $\beta = F^\circ(\beta) \rtimes \Theta \simeq \mathbb{K}_+ \rtimes \mathbb{K}^\times$

Modelling the torus

- *Observations* in PSL_2:
  Let $\iota = \begin{pmatrix} i & \\ & -i \end{pmatrix} \in \Sigma^\circ$. Note that ι inverts $F^\circ(\beta)$.
  One has $\Theta = \left\{ \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}, t \in \mathbb{K}^\times \right\} = C^\circ(\iota)$.
  Let $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \Sigma \setminus \Sigma^\circ$. Note that ω inverts Θ.
- *Modelisation* in G: for an involution $w \notin B$, let
  $$T[w] := \left\{ b \in B, b^w = b^{-1} \right\}$$
- $T[w]$ will be our model of the torus.
- Target: $B = (F^\circ(B))^{-i} \rtimes T[w]$.

Using $T[w]$

$i \in G$, $B \geq C^\circ(i)$ a Borel.
for an involution $w \notin B$, $T[w] = \left\{b \in B, b^w = b^{-1}\right\}$

- For generic w, $\mathrm{rg}\, T[w] \geq \mathrm{rg}\left(F^\circ(B)\right)^{-i}$.

Theorem (Zilber)

Let $A \rtimes T$ be a group of finite Morley rank with A, T two abelian definable infinite subgroups s.t. T is faithful and A is T-minimal.
Then there is a definable field $\mathbb{K}$ s.t. $A \simeq \mathbb{K}_+$ and $T \hookrightarrow \mathbb{K}^\times$.

- If $A \subseteq F^\circ(B)^{-i}$, ranks would force $T[w] \simeq \mathbb{K}^\times$...
- ... but $T[w]$ has no reason to be a group!
- As $T[w] \subseteq B \cap B^w$, it would be good to

control intersections of Borel subgroups

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Locally° solvable° groups

- Recall MR is no suitable notion of smallness
  (as we are unable to solve MR = 3)
- *Observation* in $(\mathrm{P})\mathrm{SL}_2$:

  if $A < G$ is infinite and abelian, $N_G^\circ(A)$ is solvable.
- Fails for finite A (e.g. $A = Z(\mathrm{SL}_2)$)
- characterizes $(\mathrm{P})\mathrm{SL}_2$ among non-solvable alg. groups

Definition

A group G is locally° solvable° if: whenever $A < G$ is infinite and abelian, $N_G^\circ(A)$ is solvable.

- Nothing to do with f.g. subgroups; follows another tradition...
- ...from finite group theory and Thompson's papers.

Results

Theorem

Let G be a locally° solvable° non-solvable connected group of finite MR.
Assume:

- $S^\circ \simeq \mathbb{Z}_{2^\infty}^d$ *with* $d \geq 1$
- *and for any involution i* $C_G^\circ(i)$ *solvable.*
- $G \not\simeq \mathrm{PSL}_2(\mathbb{K})$ *for* $\mathbb{K} \models \mathrm{ACF}_{\neq 2}$.

Then $C_G^\circ(i)$ *is always a Borel and either:*

1. $S \simeq \mathbb{Z}_{2^\infty}$
2. $S \simeq \mathbb{Z}_{2^\infty} \rtimes \langle i^g \rangle$ *and* $C^\circ(i)$ *is abelian*
3. $S \simeq \mathbb{Z}_{2^\infty}^2$ *and the three involutions are conjugate*

Complications

- Since the first counting arguments involving $T[w]$, the proofs have continuously grown more complex.
- Works by Nesin, J., Cherlin and J., D.
- Main issue: control intersections of Borel subgroups

Keywords

Here are some ingredients of a proof:

- strongly real elements and $T[w]$ sets
- $(0, d)$-Sylow subgroups
- Rigidity Lemmas
- The Bender method, Burdges' style, revisited
- concentration of semi-simple elements and contradiction!

A key observation

- Fact:

  In $(\mathrm{P})\mathrm{SL}_2$, Borel subgroups meet on tori

  (whatever that means)

- Question: can one mimic this fact in locally° solvable° groups?
- More precisely: can one prove that distinct Borel subgroups don't share unipotent elements?
- Subtelty: "unipotent elements" is non-sense to us.
  Work with unipotent *subgroups*. Define them first!

Torsion unipotence

Observation:

If $\mathbb{K} \models \mathrm{ACF_p}$, then $F^\circ(\beta) = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} = \{g \in \beta : g^p = 1\}$.

Definition

$U \leq G$ is p-unipotent if it is definable, connected, nilpotent, of exponent p^k.

Fact (Intersection control)

If G is locally$^\circ$ solvable$^\circ$ and $U \leq G$ is p-unipotent, then U lies in a unique Borel, and actually in its Fitting subgroup.

(In PSL_2, $\beta \cap \beta^\omega$ is a torus indeed, thus so is $T[\omega]$)

Burdges' unipotence

Fact (Burdges)

For each integer $d \geq 1$, there is a notion of $(0, d)$-unipotence (gradual unipotence) and a d-unipotence radical

- d is a unipotence degree (more or less heavy)
- problems
  - the d-unipotence radical is not always in the Fitting! the heaviest radical (last non-trivial) is in it.
  - Caution! two Borels can share d-unipotence.
  - two Borels of degree d can even share d-unipotence!

Rigidity Lemma

Fact (intersection control)

If G is locally° solvable° and $U \leq G$ is p-unipotent, then U is in a unique Borel, and actually in its Fitting subgroup.

Lemma

Let G be locally° solvable° and B a Borel with unipotence degree d. Let $U \lhd B$ be a $(0, d)$-unipotent subgroup. Then B is the only Borel of degree d that contains U.

- controlling the intersection $B \cap B^w$ is possible...
- ... which will enable us to split B. We're done!
- Moral: Burdges' 0-unipotence allows intersection control

Acknowledgments

Thank you!