A semi-linear group which is not affine

Pantelis E. Eleftheriou

CMAF - Universidade de Lisboa

Barcelona - November 3, 2008

Group topology

Let $\mathcal{M}$ be an o-minimal structure, and $G = \langle G, * \rangle$ a definable group, $G \subseteq M^n$.

Group topology

Let $\mathcal{M}$ be an o-minimal structure, and $G = \langle G, * \rangle$ a definable group, $G \subseteq M^n$.

Fact (Pillay, 1988)

G can be equipped with a unique definable manifold topology that makes it into a topological group. Call it G-topology.

Group topology

Let $\mathcal{M}$ be an o-minimal structure, and $G = \langle G, * \rangle$ a definable group, $G \subseteq M^n$.

Fact (Pillay, 1988)

G can be equipped with a unique definable manifold topology that makes it into a topological group. Call it G-topology.

- The G-topology and the subspace topology coincide on a large subset V of G. $\big(\dim(G \setminus V) < \dim(G)\big)$

Affine embedding

Definition

G is called *affine* if the G-topology and the subspace topology coincide on (the whole of) G. We say that G *admits an affine embedding* if there is a definable isomorphism of topological groups $\tau : G \to G' \subseteq M^r$ between G and an affine definable group G'.

Affine embedding

Definition

G is called *affine* if the G-topology and the subspace topology coincide on (the whole of) G. We say that G *admits an affine embedding* if there is a definable isomorphism of topological groups $\tau : G \to G' \subseteq M^r$ between G and an affine definable group G'.

Equivalently:

- there a definable injective map $\tau : G \to M^r$, $r \in \mathbb{N}$, which is continuous with respect to the subspace topology in the range.

Known results

G admits an affine embedding in each of the following cases:

1. (Robson, 1983) $\mathcal{M}$ is a real closed field.

Known results

G admits an affine embedding in each of the following cases:

1. (Robson, 1983) $\mathcal{M}$ is a real closed field.
2. (v. d. Dries, 1998) $\mathcal{M}$ expands a real closed field.

Known results

G admits an affine embedding in each of the following cases:

1. (Robson, 1983) $\mathcal{M}$ is a real closed field.
2. (v. d. Dries, 1998) $\mathcal{M}$ expands a real closed field.
   (Berarducci, Otero, 2001) In this case, G admits an affine embedding which is moreover a diffeomorphism.

Known results

G admits an affine embedding in each of the following cases:

1. (Robson, 1983) $\mathcal{M}$ is a real closed field.
2. (v. d. Dries, 1998) $\mathcal{M}$ expands a real closed field.
   (Berarducci, Otero, 2001) In this case, G admits an affine embedding which is moreover a diffeomorphism.
3. (Edmundo, E.) If G is semi-linear and torsion-free.

Known results

G admits an affine embedding in each of the following cases:

1. (Robson, 1983) $\mathcal{M}$ is a real closed field.
2. (v. d. Dries, 1998) $\mathcal{M}$ expands a real closed field.
   (Berarducci, Otero, 2001) In this case, G admits an affine embedding which is moreover a diffeomorphism.
3. (Edmundo, E.) If G is semi-linear and torsion-free.
   (G is definably isomorphic to $\langle M^n, +\rangle$.)

Definition (Peterzil, Steinhorn, 1999)

G is definably compact if for every definable continuous $f : (a, b) \subseteq M \to G$, $-\infty \le a, b \le \infty$, the limit $\lim^{G}_{t \to b^-} f(t)$ exists.

Definition (Peterzil, Steinhorn, 1999)

G is definably compact if for every definable continuous $f : (a, b) \subseteq M \to G$, $-\infty \leq a, b \leq \infty$, the limit $\lim^{G}_{t \to b^-} f(t)$ exists.

Fact

If $G \subseteq M^n$ is affine, then G is definably compact if and only if it is closed (in M^n) and bounded.

Semi-linear context

Let $\langle M, <, +, 0, \{d\}_{d \in D} \rangle$ be an ordered vector space over an ordered division ring D. A definable group in $\mathcal{M}$ is called *semi-linear*.

Semi-linear context

Let $\langle M, <, +, 0, \{d\}_{d\in D}\rangle$ be an ordered vector space over an ordered division ring D. A definable group in $\mathcal{M}$ is called *semi-linear*.

Fact

Every definable function $f : A \subseteq M^n \to M^m$ is piecewise-linear (PL); that is, there is a partition of A into finitely many definable sets A_i, $i = 1, \ldots, k$, such that for each of them:

- *there is an $n \times m$ matrix λ with entries from D, and an element $a \in M^m$, such that for every $x \in A_i$, $f(x) = \lambda x + a$.*

Structure Theorem

Fact (E., Starchenko, 2007)

Let G be a definably compact, definably connected semi-linear group of dimension n.

Structure Theorem

Fact (E., Starchenko, 2007)

Let G be a definably compact, definably connected semi-linear group of dimension n. Then there is a lattice L of rank n,

$$L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n \leqslant \langle M^n, + \rangle,$$

Structure Theorem

Fact (E., Starchenko, 2007)

Let G be a definably compact, definably connected semi-linear group of dimension n. Then there is a lattice L of rank n,

$$L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n \leqslant \langle M^n, + \rangle,$$

and a definable group $\langle S, +_L \rangle$, with $x +_L y = z \Leftrightarrow x + y - z \in L$,

Structure Theorem

Fact (E., Starchenko, 2007)

Let G be a definably compact, definably connected semi-linear group of dimension n. Then there is a lattice L of rank n,

$$L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n \leqslant \langle M^n, + \rangle,$$

and a definable group $\langle S, +_L \rangle$, with $x +_L y = z \Leftrightarrow x + y - z \in L$, such that

$$G \cong_{defly} \langle S, +_L \rangle.$$

Structure Theorem

Fact (E., Starchenko, 2007)

Let G be a definably compact, definably connected semi-linear group of dimension n. Then there is a lattice L of rank n,

$$L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n \leqslant \langle M^n, + \rangle,$$

and a definable group $\langle S, +_L \rangle$, with $x +_L y = z \Leftrightarrow x + y - z \in L$, such that

$$G \cong_{defly} \langle S, +_L \rangle.$$

Moreover, there is a definable "parallelogram" $H \subseteq M^n$, such that

$$G \cong < H > /L$$

Examples

Let $\mathcal{M} = \langle \mathbb{R}, <, +, 0 \rangle$.

- $G_1 = \langle [0,1), +_L, 0 \rangle$, where $L = \mathbb{Z}$.
  $x +_L y = z \Leftrightarrow x + y - z \in \mathbb{Z} \Leftrightarrow x + y - z \in \{0, 1\}$.
- $G_2 = G_1 \times G_1 = \langle [0,1) \times [0,1), +_L, 0 \rangle$, where $L = \mathbb{Z}^2$.
- $G_3 = \langle [0,1) \times [0, \pi/2)\,, +_L, 0 \rangle$, where
  $L = \mathbb{Z}(0,1) + \mathbb{Z}\,(1/2, \pi/2)$.

Let $\mathcal{M} = \langle M, <, +, 0, \{d\}_{d \in D} \rangle$ be an ordered vector space over an ordered division ring D. Define

$$x \preccurlyeq_D y \Leftrightarrow \exists d \in D, |x| \leq d|y|.$$

Let $\mathcal{M} = \langle M, <, +, 0, \{d\}_{d \in D} \rangle$ be an ordered vector space over an ordered division ring D. Define

$$x \preccurlyeq_D y \Leftrightarrow \exists d \in D, |x| \leq d|y|.$$

We let $x \prec_D y$ if $x \preccurlyeq_D y$ but not $y \preccurlyeq_D x$.

Let $\mathcal{M} = \langle M, <, +, 0, \{d\}_{d \in D} \rangle$ be an ordered vector space over an ordered division ring D. Define

$$x \preccurlyeq_D y \;\Leftrightarrow\; \exists d \in D, |x| \leq d|y|.$$

We let $x \prec_D y$ if $x \preccurlyeq_D y$ but not $y \preccurlyeq_D x$. We have:

$$x \prec_D y \;\Leftrightarrow\; \forall d \in D,\ d|x| < |y|.$$

Counterexample

Example

- Let $\mathcal{M} = \langle M, +, <, 0 \rangle$ be an ordered divisible abelian group.

Counterexample

Example

- Let $\mathcal{M} = \langle M, +, <, 0 \rangle$ be an ordered divisible abelian group.
- Let $a, b, c > 0$ in M, such that $b \prec_{\mathbb{Q}} c \prec_{\mathbb{Q}} a$.

Counterexample

Example

- Let $\mathcal{M} = \langle M, +, <, 0 \rangle$ be an ordered divisible abelian group.
- Let $a, b, c > 0$ in M, such that $b \prec_{\mathbb{Q}} c \prec_{\mathbb{Q}} a$. In particular,
  - $\nexists$ definable onto $f : [0, b) \to [0, c)$, and
  - $\forall n \in \mathbb{N}$, $nc < a$.

Counterexample

Example

- Let $\mathcal{M} = \langle M, +, <, 0 \rangle$ be an ordered divisible abelian group.
- Let $a, b, c > 0$ in M, such that $b \prec_{\mathbb{Q}} c \prec_{\mathbb{Q}} a$. In particular,
  - $\nexists$ definable onto $f : [0, b) \longrightarrow [0, c)$, and
  - $\forall n \in \mathbb{N}$, $nc < a$.
- Let $G = \langle [0, a) \times [0, b), +_L, 0 \rangle$, where $L = \mathbb{Z}(a, 0) + \mathbb{Z}(a - c, b) \leqslant M^2$.

Counterexample

Example

- Let $\mathcal{M} = \langle M, +, <, 0\rangle$ be an ordered divisible abelian group.
- Let $a, b, c > 0$ in M, such that $b \prec_{\mathbb{Q}} c \prec_{\mathbb{Q}} a$. In particular,
  - $\nexists$ definable onto $f : [0, b) \to [0, c)$, and
  - $\forall n \in \mathbb{N}$, $nc < a$.
- Let $G = \langle [0, a) \times [0, b), +_L, 0\rangle$, where $L = \mathbb{Z}(a, 0) + \mathbb{Z}(a - c, b) \leqslant M^2$.

Assume that $\tau : G \to M^r$ is an affine embedding.

For every $t \in [0, a - c)$, consider the one-to-one G-path

- $\phi_t : [0, b) \to \{t\} \times [0, b)$, with $\phi_t(x) = (t, x)$.

For every $t \in [0, a - c)$, consider the one-to-one G-path

- $\phi_t : [0, b) \to \{t\} \times [0, b)$, with $\phi_t(x) = (t, x)$.

For every $t \in [0, a - c]$,

- $\lim^G_{x \to b} \phi_t(x) = (t + c, 0)$

For every $t \in [0, a - c)$, consider the one-to-one G-path

- $\phi_t : [0, b) \to \{t\} \times [0, b)$, with $\phi_t(x) = (t, x)$.

For every $t \in [0, a - c]$,

- $\lim^G_{x \to b} \phi_t(x) = (t + c, 0)$
- $\lim_{x \to b} \tau\big(\phi_t(x)\big) = \tau(t + c, 0)$

For every $t \in [0, a - c)$, consider the one-to-one G-path

- $\phi_t : [0, b) \to \{t\} \times [0, b)$, with $\phi_t(x) = (t, x)$.

For every $t \in [0, a - c]$,

- $\lim^G_{x \to b} \phi_t(x) = (t + c, 0)$
- $\lim_{x \to b} \tau\big(\phi_t(x)\big) = \tau(t + c, 0)$

Since $\forall n \in \mathbb{N}$, $nc < a$, and τ is piecewise linear on $[0, a)$,

For every $t \in [0, a-c)$, consider the one-to-one G-path

- $\phi_t : [0, b) \to \{t\} \times [0, b)$, with $\phi_t(x) = (t, x)$.

For every $t \in [0, a-c]$,

- $\lim^G_{x \to b} \phi_t(x) = (t+c, 0)$
- $\lim_{x \to b} \tau\big(\phi_t(x)\big) = \tau(t+c, 0)$

Since $\forall n \in \mathbb{N}$, $nc < a$, and τ is piecewise linear on $[0, a)$,

- $\exists n \in \mathbb{N}$ such that τ is linear on $\big[nc, (n+1)c\big)$.

For every $t \in [0, a-c)$, consider the one-to-one G-path

- $\phi_t : [0, b) \to \{t\} \times [0, b)$, with $\phi_t(x) = (t, x)$.

For every $t \in [0, a-c]$,

- $\lim^G_{x \to b} \phi_t(x) = (t + c, 0)$
- $\lim_{x \to b} \tau\big(\phi_t(x)\big) = \tau(t + c, 0)$

Since $\forall n \in \mathbb{N}$, $nc < a$, and τ is piecewise linear on $[0, a)$,

- $\exists n \in \mathbb{N}$ such that τ is linear on $\big[nc, (n+1)c\big)$.

$\tau(\phi_{nc}) : [0, b) \to M^r$ has endpoints $\tau(nc, 0)$ and $\tau\big((n+1)c, 0\big)$,

- $\exists$ definable onto $f : \tau(\phi_{nc})\big([0, b)\big) \to \tau\Big(\big[nc, (n+1)c\big) \times \{0\}\Big)$.

For every $t \in [0, a-c)$, consider the one-to-one G-path

- $\phi_t : [0, b) \to \{t\} \times [0, b)$, with $\phi_t(x) = (t, x)$.

For every $t \in [0, a-c]$,

- $\lim^G_{x \to b} \phi_t(x) = (t + c, 0)$
- $\lim_{x \to b} \tau\big(\phi_t(x)\big) = \tau(t + c, 0)$

Since $\forall n \in \mathbb{N}$, $nc < a$, and τ is piecewise linear on $[0, a)$,

- $\exists n \in \mathbb{N}$ such that τ is linear on $\big[nc, (n+1)c\big)$.

$\tau(\phi_{nc}) : [0, b) \to M^r$ has endpoints $\tau(nc, 0)$ and $\tau\big((n+1)c, 0\big)$,

- $\exists$ definable onto $f : \tau(\phi_{nc})\big([0, b)\big) \to \tau\Big(\big[nc, (n+1)c\big) \times \{0\}\Big)$.
- Hence, $\exists$ definable onto $f : [0, b) \to [0, c)$,

For every $t \in [0, a-c)$, consider the one-to-one G-path

- $\phi_t : [0, b) \to \{t\} \times [0, b)$, with $\phi_t(x) = (t, x)$.

For every $t \in [0, a-c]$,

- $\lim^G_{x \to b} \phi_t(x) = (t + c, 0)$
- $\lim_{x \to b} \tau\big(\phi_t(x)\big) = \tau(t + c, 0)$

Since $\forall n \in \mathbb{N}$, $nc < a$, and τ is piecewise linear on $[0, a)$,

- $\exists n \in \mathbb{N}$ such that τ is linear on $\big[nc, (n+1)c\big)$.

$\tau(\phi_{nc}) : [0, b) \to M^r$ has endpoints $\tau(nc, 0)$ and $\tau\big((n+1)c, 0\big)$,

- $\exists$ definable onto $f : \tau(\phi_{nc})\big([0, b)\big) \to \tau\Big(\big[nc, (n+1)c\big) \times \{0\}\Big)$.
- Hence, $\exists$ definable onto $f : [0, b) \to [0, c)$,

a contradiction.

Theorem

Let $\mathcal{M} = \langle M, +, <, 0 \rangle$ *be an ordered divisible abelian group.*
Let $G = \langle [0, a) \times [0, b), +_L, 0 \rangle$*, where*
$L = \mathbb{Z}(a, 0) + \mathbb{Z}(a - c, b) \leqslant M^2$*.*

Theorem

Let $\mathcal{M} = \langle M, +, <, 0\rangle$ *be an ordered divisible abelian group.*
Let $G = \langle [0, a) \times [0, b), +_L, 0\rangle$*, where*
$L = \mathbb{Z}(a, 0) + \mathbb{Z}(a - c, b) \leqslant M^2$*.*

Then G *admits an affine embedding if and only if*

- $\exists m, n \in \mathbb{Z},\ m^2 + n^2 \neq 0,\ ma + nc \preccurlyeq_D b.$

Let $\langle M, <, +, 0, \{d\}_{d \in D} \rangle$ be an ordered vector space over an ordered division ring D. Let G be a definably compact, definably connected semi-linear group of dimension n.

Let $\langle M, <, +, 0, \{d\}_{d \in D} \rangle$ be an ordered vector space over an ordered division ring D. Let G be a definably compact, definably connected semi-linear group of dimension n. By the Structure Theorem:

$$G \cong_{defly} \langle S, +_L \rangle \cong < H > /L,$$

where

- $L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n \leqslant \langle M^n, + \rangle$,
- $H = \{\lambda_1 t_1 + \cdots + \lambda_n t_n : -e_i < t_i < e_i\}$, with $e_i > 0$ in M, and $\lambda_i \in D^n$.

Let $\langle M, <, +, 0, \{d\}_{d \in D} \rangle$ be an ordered vector space over an ordered division ring D. Let G be a definably compact, definably connected semi-linear group of dimension n. By the Structure Theorem:

$$G \cong_{defly} \langle S, +_L \rangle \cong < H > /L,$$

where

- $L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n \leqslant \langle M^n, + \rangle$,
- $H = \{\lambda_1 t_1 + \cdots + \lambda_n t_n : -e_i < t_i < e_i\}$, with $e_i > 0$ in M, and $\lambda_i \in D^n$.

Up to a linear transformation, we may assume that H is a rectangle.

Theorem

Let $G =< H > /L$ *be a definably compact, definably connected semi-linear group of dimension* n*, where*

- $L = \mathbb{Z}(a_1, a_2) + \mathbb{Z}(b_1, b_2) \leqslant \langle M^2, + \rangle$
- H *is a rectangle.*

Theorem

Let $G = < H > / L$ be a definably compact, definably connected semi-linear group of dimension n, where

- $L = \mathbb{Z}(a_1, a_2) + \mathbb{Z}(b_1, b_2) \leqslant \langle M^2, + \rangle$
- *H is a rectangle.*

Then, G admits an affine embedding if and only if the following two conditions both hold:

1. $\exists m_1, n_1 \in \mathbb{Z},\ m_1^2 + n_1^2 \neq 0,\ m_1 a_1 + n_1 b_1 \preccurlyeq_D |a_2| + |b_2|,$
2. $\exists m_2, n_2 \in \mathbb{Z},\ m_2^2 + n_2^2 \neq 0,\ m_2 a_2 + n_2 b_2 \preccurlyeq_D |a_1| + |b_1|.$

Theorem

Let $G = < H > /L$ be a definably compact, definably connected semi-linear group of dimension n, where

- $L = \mathbb{Z}(a_1, a_2) + \mathbb{Z}(b_1, b_2) \leqslant \langle M^2, + \rangle$
- *H is a rectangle.*

Then, G admits an affine embedding if and only if the following two conditions both hold:

1. $\exists m_1, n_1 \in \mathbb{Z},\ m_1^2 + n_1^2 \neq 0,\ m_1 a_1 + n_1 b_1 \preccurlyeq_D |a_2| + |b_2|,$
2. $\exists m_2, n_2 \in \mathbb{Z},\ m_2^2 + n_2^2 \neq 0,\ m_2 a_2 + n_2 b_2 \preccurlyeq_D |a_1| + |b_1|.$

Corollary

If $\mathcal{M}$ is Archimedean, then G admits an affine embedding.

Classical PL-topology

- (Whitney, 1944) Every real PL-manifold of dimension n admits an affine embedding into $\mathbb{R}^{2n}$.

Classical PL-topology

- (Whitney, 1944) Every real PL-manifold of dimension n admits an affine embedding into $\mathbb{R}^{2n}$.
- (Burago, Zalgaller, 1995) Every orientable real PL-manifold of dimension 2 admits an isometric affine embedding into $\mathbb{R}^3$. The generalization of this statement to manifolds of higher dimension is open.