Combinatorial Geometries of the Hrushovski Constructions

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(1.1) Strongly minimal structures

An infinite L-structure D is strongly minimal if every definable subset of D is finite or cofinite in D, uniformly in the defining formula: for every L-formula $\varphi(x, \bar{y})$ there is n_φ such that for all parameters $\bar{a}$ either $\{c \in D : D \models \varphi(c, \bar{a})\}$ or its complement in D has at most n_φ elements.

EXAMPLES:

1. Pure set $(S; =)$
2. K-vector space $(V; +, 0, (\lambda_s : s \in K))$; K any division ring
3. Algebraically closed field $(F; +, -, \cdot, 0, 1)$
4. D_μ : Hrushovski's 3-ary structures from 1988 (published in 1993).
5. Fusions
6. ... ?

(1.2) Algebraic closure

In any structure M, if $X \subseteq M$ define the algebraic closure $\mathrm{acl}(X)$ of X in M to be the union of the finite X-definable subsets of M.

This is a (good) closure operator on M, and if M is strongly minimal, then it satisfies the exchange property, giving us a pregeometry.

(1.3) Pregeometries

Suppose A is any set; denote by $\mathcal{P}(A)$ the power set of A. A function $\mathrm{cl} : \mathcal{P}(A) \to \mathcal{P}(A)$ is a closure operation on A if for all $X \subseteq Y \subseteq A$:

- $X \subseteq \mathrm{cl}(X)$
- $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$
- $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$
- $\mathrm{cl}(X) = \bigcup\{\mathrm{cl}(X_0) : X_0 \subseteq X \text{ finite }\}$.

We say that (A, cl) is a pregeometry if additionally it satisfies:

- (Exchange) If $a \in \mathrm{cl}(X \cup \{b\}) \setminus \mathrm{cl}(X)$ then $b \in \mathrm{cl}(X \cup \{a\})$.

Suppose $X \subseteq Y \subseteq A$. Say that X is an independent set if $a \notin \mathrm{cl}(X \setminus \{a\})$ for all $a \in X$. If also $\mathrm{cl}(X) = \mathrm{cl}(Y)$, say that X is a basis of Y. Then we have:

- Any subset Y of A has a basis;
- Any two bases of Y have the same cardinality, called the dimension of Y.

Geometries

A pregeometry (B, cl) is a geometry if it satisfies

- $\mathrm{cl}(b) = \{b\}$ for all $b \in B$.

Given a pregeometry (A, cl) the relation

$$a \sim b \Leftrightarrow \mathrm{cl}(a) = \mathrm{cl}(b)$$

is an equivalence relation on $A \setminus \mathrm{cl}(\emptyset)$. The set $\tilde{A}$ of equivalence classes inherits a closure operation $\tilde{\mathrm{cl}}$ and $(\tilde{A}, \tilde{\mathrm{cl}})$ is a geometry with whose lattice of closed sets is naturally isomorphic to that of the pregeometry (A, cl).

If $X \subseteq A$ the localization of (A, cl) at X is the pregeometry on A with closure $\mathrm{cl}_X(Y) = \mathrm{cl}(Y \cup X)$. The geometry of the localization has lattice of closed sets isomorphic to the lattice of closed sets in (A, cl) which contain $\mathrm{cl}(X)$.

(1.4) Examples from sm structures

Look at the geometry arising from algebraic closure in the examples of sm structures:

- Pure set $(S; =)$. Here $\mathrm{cl}(X) = X$: the geometry is disintegrated.
- K-Vector space $(V; +, 0, (\lambda_s : s \in K))$: cl is linear closure and the geometry is the projective geometry $\mathbb{P}(V)$.
- Algebraically closed field $(F; +, \cdot, (c_e : e \in E))$, E a subfield. cl is algebraic closure over E; denote the geometry by $\mathcal{G}(F/E)$.
- Hrushovski examples D_μ: Study this.

(1.5) Other examples of geometries from model theory

Arise from forking on a regular type.

EXAMPLE: In a model of DCF_0, take the closure operation of differential dependence.

(1.6) Recovering the structure from the geometry

1. If $dim_K(V) \geq 3$ the Fundamental Theorem of Projective Geometry uniformly interprets K and V in $\mathbb{P}(V)$.
2. If $F \supseteq E$ are algebraically closed and $trdeg(F/E) \geq 5$ then F and E can be uniformly interpreted in $\mathcal{G}(F/E)$ (DE + E. Hrushovski, 1995).
3. Generalization of this where F, E not assumed algebraically closed (J. Gismatullin, 2008).
4. If $F \models DCF_0$ is saturated then the pure field F can be uniformly interpreted in the geometry of differential dependence on F and any automorphism of the geometry arises from a field automorphism which preserves differential dependence (R. Konnerth, 2002).

QUESTION: What happens with the D_μ?

(2.1) Predimension

Language L: 3-ary relation symbol R.
If A is an L-structure the corresponding relation in A is $R^A \subseteq A^3$.
For a finite L-structure B the predimension of B is

$$\delta(B) = |B| - |R^B|.$$

For $A \subseteq B$ say that A is self-sufficient in B and write $A \leq B$ if

$$\delta(A) \leq \delta(B') \text{ for all } B' \text{ with } A \subseteq B' \subseteq B.$$

Properties:

- $A \leq B$ and $X \subseteq B \Rightarrow X \cap A \leq X$
- $A \leq B \leq C \;\Rightarrow\; A \leq C$
- Self-sufficient closure: $\mathrm{cl}_B^{\leq}(X) := \bigcap\{A : X \subseteq A \leq B\} \leq B$

Extend to arbitrary L-structures $A \subseteq B$ by:

$$A \leq B \Leftrightarrow X \cap A \leq X \text{ for all finite } X \subseteq B.$$

(2.2) Dimension

Let $\bar{\mathcal{C}}$ be the class of L-structures A with $\emptyset \leq A$: so $\delta(X) \geq 0$ for all finite $X \subseteq A$. Let $\mathcal{C}$ be the finite structures in $\bar{\mathcal{C}}$.

If X is a finite subset of $B \in \bar{\mathcal{C}}$ there is a finite Y with $X \subseteq Y \subseteq B$ and $\delta(Y)$ as small as possible. Then $Y \leq B$ and so $\mathrm{cl}^{\leq}_B(X) \subseteq Y$ is finite.

The dimension of X in B is:

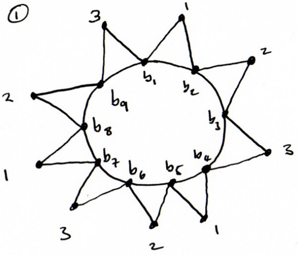
$$d_B(X) = \delta(\mathrm{cl}^{\leq}_B(X)).$$

The d-closure of X in B is:

$$\mathrm{cl}^d_B(X) = \{a \in B : d_B(X \cup \{a\}) = d_B(X)\}.$$

FACT: (B, cl^d_B) is a pregeometry. Dimension in the pregeometry is d_B.

Examples

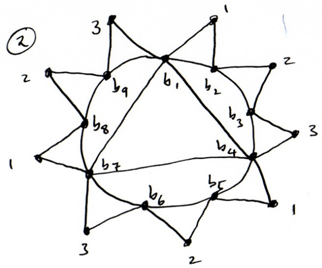


$A = \{1,2,3\}$

$B = \{1,2,3, b_1, \ldots, b_9\}$

$\delta(A) = \delta(B)$

$A \leq B$. $\quad B = cl^d_B(A)$



$A = \{1,2,3\}$

$B = \{1,2,3, b_1, \ldots, b_9\}$

$\delta(B) = \delta(A) - 1$

$B = cl^{\leq}_B(A)$

(2.3) Free amalgamation and the generic structure

If $B_1, B_2 \in \bar{\mathcal{C}}$ have a common substructure A, the free amalgam

$$B_1 \coprod_A B_2$$

of B_1 and B_2 over A is the structure whose domain is the disjoint union of B_1 and B_2 over A and whose relations are just those of B_1 and B_2.

EASY AMALGAMATION LEMMA: If $A \leq B_1$ then $B_2 \leq B_1 \coprod_A B_2 \in \bar{\mathcal{C}}$.

So $(\mathcal{C}, \leq)$ is an amalgamation class.

COROLLARY: There is a countable $M_3 \in \bar{\mathcal{C}}$ with the property that whenever $A \leq M_3$ is finite and $A \leq B \in \mathcal{C}$ then there exists an embedding $f : B \to M_3$ with $f(a) = a$ for all $a \in A$ and $f(B) \leq M_3$. This property determines M_3 up to isomorphism amongst countable structures in $\bar{\mathcal{C}}$ and any isomorphism between finite $\leq$-substructures of M_3 extends to an automorphism of M_3.

(2.4) Properties of the generic structure

The structure M_3 is called the generic structure associated to the amalgamation class $(\mathcal{C}, \leq)$.

FACTS:

- M_3 is ω-stable of MR ω
- algebraic closure in M_3 is equal to self-sufficient closure and does not satisfy exchange
- (M_3, cl^d) is a pregeometry; denote the corresponding geometry by $\mathcal{G}(M_3)$.
- there is a unique 1-type of rank ω: points of d-dimension 1 in M_3.

(2.5) Some results

We can repeat the construction with a 4-ary relation and obtain a generic structure M_4 and compare the resulting geometries.

THEOREM A (Marco Ferreira, 2007)

The following hold:

1. $\mathcal{G}(M_3)$ is not isomorphic to $\mathcal{G}(M_4)$;
2. $\mathcal{G}(M_3)$ and $\mathcal{G}(M_4)$ have the same finite subgeometries;
3. $\mathcal{G}(M_3)$ is isomorphic to any of its localizations over a finite set.

In fact the same is true replacing $3, 4$ here by any $m \neq n$. There is also a statement about generic structures constructed using a predimension of the form

$$|A| - \sum_{i \in I} |R_i^A|$$

where the R_i are relations of varying arities.

(3.1) The Amalgamation class $(\mathcal{C}_\mu, \leq)$

Want a similar construction where *d*-closure is equal to algebraic closure ('collapse').

Keep the class $\mathcal{C}$, the predimension δ, the notion of self-sufficient embedding $\leq$ from the previous section.

DEFINITION: A pair of structures $A \leq B \in \mathcal{C}$ with $A \neq B$ is a

- algebraic extension if $\delta(A) = \delta(B)$
- simple algebraic extension if also $\delta(A) < \delta(B')$ whenever $A \subset B' \subset B$
- minimal simple algebraic extension if also for every $A' \subset A$ the extension $A' \subseteq A' \cup (B \setminus A)$ is not simply algebraic.

Now fix a function μ from the class of isomorphism types of msa extensions to $\mathbb{N}$ such that for each msa $A \leq B$ we have

$$\mu(A, B) \geq \delta(A).$$

DEFINITION: The class $\mathcal{C}_\mu$ consists of all structures X in $\mathcal{C}$ which for every msa $A \leq B$ omit $\mu(A, B) + 1$ copies of B over A. More precisely, if $B_1, \ldots, B_n \subseteq X$ have pairwise intersection A_0 and (A_0, B_i) is isomorphic to (A, B) for each $i \leq n$, then $n \leq \mu(A, B)$.

THEOREM (Ehud Hrushovski, 1993)

- The class $(\mathcal{C}_\mu, \leq)$ is an amalgamation class.
- There is a (unique) countable structure $D_\mu \in \bar{\mathcal{C}}_\mu$ with the property that whenever $A \leq D_\mu$ is finite and $A \leq B \in \mathcal{C}_\mu$, there is an embedding $f : B \to D_\mu$ with $f(a) = a$ for all $a \in A$ and $f(B) \leq D_\mu$.
- Algebraic closure in D_μ is equal to d-closure.
- D_μ is strongly minimal.

– Get continuum many non-isomorphic strongly minimal structures by varying μ.

(3.2) Geometry of the D_μ

Theorem B (Marco Ferreira, 2008)

The geometry $\mathcal{G}(D_\mu)$ of algebraic closure in D_μ is isomorphic to the geometry $\mathcal{G}(M_3)$ of d-closure in the 'uncollapsed' M_3.

(3.3) Questions

1. What about the geometries of other models of $Th(D_\mu)$ and $Th(M_3)$ and localizations over infinite subsets?
2. There is a variation on the construction, again due to Hrushovski, which produces sm sets D'_μ where the algebraic closure of a pair of points has size 3: non-isomorphic structures give non-isomorphic geometries. Are the localizations of these geometries (over, say a 2-dimensional set) isomorphic to $\mathcal{G}(M_3)$?

(4.1) Methods of proof: Theorem A

3-ary language; take δ, $(\mathcal{C}, \leq)$, M_3 as before.

IDEA: Given $B \in \bar{\mathcal{C}}$, change the structure on some finite $A \leq B$ to $A' \in \mathcal{C}$ (– same set, different structure). This gives a new structure B' with the same underlying set as B.

Changing Lemmas

1. $A' \leq B'$ and $B' \in \bar{\mathcal{C}}$.
2. If $B = M_3$ then $B' \cong M_3$.
3. If d-closure is the same in A and A' then it is the same in B and B'.
4. If $d(A') = 0$ then the pregeometry on B' is the localization of B over A.

A similar result holds for n-ary structures.

(4.2) Embedding pregeometries

For $A \in \mathcal{C}$ let $\mathcal{PG}(A)$ denote the pregeometry (A, cl^d_A). Let $\mathcal{P}$ be the resulting class of pregeometries. Make this into a functor:

$$(\mathcal{C}, \leq) \overset{\mathcal{PG}}{\rightsquigarrow} (\mathcal{P}, \preceq).$$

Thus for $A \subseteq B \in \mathcal{P}$ we have $A \preceq B$ iff there are structures $\tilde{A} \leq \tilde{B} \in \mathcal{C}$ with underlying sets A, B whose d-closure gives the pregeometry on B.

Theorem C

1. $(\mathcal{P}, \preceq)$ is an amalgamation class.
2. The pregeometry which is the generic structure of this class is isomorphic to $\mathcal{PG}(M_3)$.

Similar results hold for n-ary structures.

(4.3) Proof of Theorem B

The Changing Lemma fails for $\mathcal{C}_\mu$. Instead we have:

Hard Changing Lemma

Suppose $A \leq B \in \mathcal{C}$ and $A \in \mathcal{C}_\mu$. Then there is $B' \in \mathcal{C}_\mu$ with

$$A \leq B' \text{ and } \mathcal{PG}(B) \preceq \mathcal{PG}(B').$$

REMARKS:

- Cannot take B a substructure of B' here.
- Together with the Changing Lemmas for M_3, this allows us to build an isomorphism $\mathcal{PG}(M_3) \cong \mathcal{PG}(D_\mu)$ by back and forth.
- Result *should* hold for n-ary structures, but the details are hard.