Trees of definable sets in $\mathbb{Z}_p$

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Setting

- Work in $\mathbb{Q}_p$ (p-adics)
- Two-sorted language:
  - $\mathbb{Q}_p$ with field language
  - value group $\mathbb{Z}$ with ordered group language
  - valuation $v : \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}$
- For tuples $\bar{x} \subset \mathbb{Q}_p^n$, set $v(\bar{x}) := \min\{v(x_i) \mid 1 \leq i \leq n\}$
- **Ball** around $\bar{x} \in \mathbb{Q}_p^n$ of (valuative) radius $\lambda \in \mathbb{Z}$:
  $B(\bar{x}, \lambda) := \{\bar{x}' \mid v(\bar{x}' - \bar{x}) \geq \lambda\} = \bar{x} + p^\lambda \mathbb{Z}_p^n$
- Recall: Balls of fixed radius λ form a partition of $\mathbb{Q}_p^n$

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Trees of sets in $\mathbb{Z}_p^n$

- Subballs of $\mathbb{Z}_p^n$ form a tree
- Suppose $\emptyset \neq X \subset \mathbb{Z}_p^n$. This yields a sub-tree **T(X)** of those balls intersecting X:
  - Vertices: $\mathrm{T}(X) = \{B = B(\bar{x}, \lambda) \mid \bar{x} \in X, \lambda \geq 0\}$
  - Root of $\mathrm{T}(X)$ is $\mathbb{Z}_p$
  - Tree structure given by inclusion
- $B \subset \mathbb{Z}_p^n$ ball, $B \cap X \neq \emptyset$
  $\rightsquigarrow$ $\mathbf{T_B(X)}$:= subtree of $\mathrm{T}(X)$ above B (i.e. vertices are balls contained in B)

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Examples of trees

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  - $X = \mathbb{Z}_p^n \rightsquigarrow$ Every node of $\mathrm{T}(X)$ has p^n children
  - X finite $\rightsquigarrow$ each $x \in X$ corresponds to infinite path in $\mathrm{T}(X)$...
- **Infinite path** in a tree $\mathcal{T}$ is a sequence of vertices $\{v_0, v_1, v_2, \dots\} \subset \mathcal{T}$ with $v_0 =$ root and v_{i+1} is child of v_i
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- Obvious condition: $\mathcal{T}$ has no leaves.
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  - This yields $x \in \bar{X}$.
  - Define: **Side branch of $\mathcal{P}$** := subtree of $\mathcal{T}$ starting at a vertex of $\mathcal{P}$, without $\mathcal{P}$ itself.
  - $\mathcal{P}$ has a side branch at depth $\lambda \iff$ there exists $x' \in X$ such that $v(x' - x) = \lambda$
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Motivation: isometry

- $\mathrm{T}(X) = \mathrm{T}(\bar{X})$; so **from now on suppose $\mathbf{X = \bar{X}}$**

Lemma

$X \subset \mathbb{Z}_p^n, X' \subset \mathbb{Z}_p^{n'}$ Then:

$\{\text{bijective isometries } X \to X'\} \stackrel{1:1}{\longleftrightarrow} \{\mathrm{T}(X) \xrightarrow{\sim} \mathrm{T}(X')\}$

Sketch of proof:

- $\phi\colon X \to X'$ yields $\mathrm{T}(X) \to \mathrm{T}(X')$, $B(\bar{x}, \lambda) \mapsto B(\phi(\bar{x}), \lambda)$ for $\bar{x} \in X$. Well-defined as ϕ isometry.
- $\psi\colon \mathrm{T}(X) \to \mathrm{T}(X')$ induces map on infinite paths... □

Analogously: $\{\text{bij. isomet. } X \cap B \to X' \cap B\} \stackrel{1:1}{\leftrightarrow} \{\mathrm{T}_B(X) \xrightarrow{\sim} \mathrm{T}_B(X')\}$

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The conjecture

- For $X \subset \mathbb{Q}_p^n$ definable, Scowcroft and van den Dries defined: **dim X** := dimension of Zariski closure of X in $\tilde{\mathbb{Q}}_p^n$.
- We will define **trees of level d** (purely combinatorial).

Conjecture (H.)

Suppose $\mathcal{T}$ is an (abstract) tree. Then:

$\mathcal{T}$ *is of level* d $\iff$ *there exists definable* $X \subset \mathbb{Z}_p^n$ *with* $\dim X = d$ *such that* $\mathcal{T} \cong \mathrm{T}(X)$

Theorem (H.)

"$\Longrightarrow$" holds.

"$\Longleftarrow$" holds if $X \subset \mathbb{Z}_p^2$ or if $\dim X \leq 1$ or if X is algebraic without singularities.

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The conjecture

- For $X \subset \mathbb{Q}_p^n$ definable, Scowcroft and van den Dries defined: **dim X** $:=$ dimension of Zariski closure of X in $\tilde{\mathbb{Q}}_p^n$.
- We will define **trees of level d** (purely combinatorial).

Conjecture (H.)

Suppose $\mathcal{T}$ is an (abstract) tree. Then:

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Definition of level 0 trees

- Definition of level d trees is recursive. Start with level 0:
  - Define: $\mathcal{T}$ **is of level 0** $\iff$ $\mathcal{T}$ has no leaves and only finitely many bifurcations.
  - These are exactly the trees of finite sets.
    $\Rightarrow$ Conjecture holds for $d = 0$:
- For $d \geq 1$ we will define when $\mathcal{T}$ is of level $\leq$ **d**
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Meaning of ❶ for the definable set

What does "T(X) satisfies ❶" mean for X?

Lemma

$X \subset \mathbb{Z}_p^n$. (Recall: X closed!) Then $\mathrm{T}(X)$ satisfies ❶ $\iff$
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Around each $\bar{x} \in X \setminus X_0$ there exists ball $B_{\bar{x}}$
such that $\mathrm{T}_{B_{\bar{x}}}(X) \cong \mathcal{T}' \times \mathrm{T}(\mathbb{Z}_p)$, where $\mathcal{T}'$ of level $\leq d-1$

Idea of proof:

- X_0 = set of points corresponding to paths $\mathcal{S}_0$
- "$\Longrightarrow$": easy.
- "$\Longleftarrow$": use that X is compact. □

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We prove that $\mathrm{T}(X)$ is of level $\leq \dim X$ in some special cases.

- $X = \{(x, \phi(x)) \mid x \in \mathbb{Z}_p\}$ with $v(\phi(x_1) - \phi(x_2)) \geq v(x_1 - x_2)$ (*)
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Conjecture yields Stratifications

Suppose the conjecture holds. Then definable sets X can be *stratified*, i.e. decomposed into nice subsets:

- $X_0 :=$ set corresponding to paths $\mathcal{S}_0$ in $\mathrm{T}(X)$
  $X_0 \subset X$ is set of "0-dimensional singularities"
- $X \setminus X_0$ can be partitioned into $X \cap B$ for balls B, such that $\mathrm{T}_B(X) \cong \mathcal{T}' \times \mathrm{T}(\mathbb{Z}_p)$ with $\mathcal{T}'$ of level $\leq d-1$
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- Inductively obtain $X = X_0 \,\dot\cup \ldots \dot\cup\, X_d$
  This is a *stratification*:
  - X_i is locally isometric to $\mathbb{Z}_p^i$ near each $\bar{x} \in X_i$
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