Raising to generic powers

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Abstract

We prove unconditionally a Schanuel property for raising to a generic real power, leading to the hope that the real field with a generic power function can be proved to be decidable. This is joint work with A.J. Wilkie and Martin Bays.

Outline

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1. Motivation – Decidability
2. Schanuel Properties
3. Proofs

Decidability of $\mathbb{R}$

Theorem (Tarski 1949)

The theory of the real field $\langle \mathbb{R}; +, \cdot \rangle$ is decidable.

Proof uses:

- model completeness
- A decision procedure for $\exists$-sentences

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Decidability of $\mathbb{R}_{\exp}$

Theorem (Wilkie 1996)

$\mathbb{R}_{\exp}$ is model-complete and o-minimal

Theorem (Macintyre, Wilkie 1996)

Assuming Schanuel's Conjecture, there is a decision procedure for $\exists$-sentences.

Corollary

Conditionally, $\mathbb{R}_{\exp}$ is decidable.

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Decidability of other functions

Question (Jones, et al.)

Can we unconditionally prove decidability for $\langle \mathbb{R}; +, \cdot, f \rangle$ *for some (interesting) analytic function* f*?*

Raising to a power $\lambda \in \mathbb{R}$

For $y > 0$, $y^\lambda = \exp(\lambda \log y)$

$\mathbb{R}_\lambda = \langle \mathbb{R}; +, \cdot, (-)^\lambda \rangle$ is o-minimal and model complete (Wilkie / Miller)

Work in progress – Jones, Servi

- Schanuel Property for $(-)^\lambda \implies$ decidability of $\mathbb{R}_\lambda$
- (if λ is a recursive real – otherwise decidability modulo λ)

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Schanuel Property for raising to a power

Theorem (Bays, Kirby, Wilkie)

Let $\lambda \in \mathbb{R}$ be exponentially transcendental, let $\bar{y} \in (\mathbb{R}_{>0})^n$, and suppose $\bar{y}$ is multiplicatively independent. Then

$$\operatorname{td} \mathbb{Q}(\bar{y}, \bar{y}^{\lambda}, \lambda)/\mathbb{Q}(\lambda) \geqslant n.$$

- λ is exponentially transcendental iff it does not lie in the prime model of $\mathbb{R}_{\exp}$.
- Co-countably many reals are exponentially transcendental.
- No known exponentially transcendental reals!
- Cantor's argument gives a recursive exponentially transcendental real.

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Exponential Transcendence - general definition

Let $\langle F; +, \cdot, \exp\rangle$ be any exponential field.

Definition

$x \in F$ is exponentially algebraic in F iff for some $n \in \mathbb{N}$ there are:

- $\bar{x} = (x_1, \ldots, x_n) \in F^n$
- $f_1, \ldots, f_n \in \mathbb{Z}[\bar{X}, e^{\bar{X}}]$

such that

- $x = x_1$
- $f_i(\bar{x}, e^{\bar{x}}) = 0$ for each $i = 1, \ldots, n$
- $\begin{vmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial X_1} & \cdots & \frac{\partial f_n}{\partial X_n} \end{vmatrix}(\bar{x}) \neq 0$

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A generalization

First Theorem

Let $\lambda \in \mathbb{R}$ be exponentially transcendental, let $\bar{y} \in (\mathbb{R}_{>0})^n$, and suppose $\bar{y}$ is multiplicatively independent. Then

$$\operatorname{td}(\mathbb{Q}(\bar{y}, \bar{y}^\lambda, \lambda)/\mathbb{Q}(\lambda)) \geqslant n.$$

Theorem (BKW)

F any exponential field, $\lambda \in F$ exponentially transcendental, $\bar{x} \in F^n$ such that $\exp(\bar{x})$ is multiplicatively independent. Then

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Several powers

For $A \subseteq F$, can define ecl A, the exponential algebraic closure of A.

Theorem (Kirby)

ecl is a pregeometry on any exponential field. Thus we have a notion of independence.

Theorem (BKW – question of Zilber)

Let $\lambda_1, \ldots, \lambda_m$ be ecl-independent in F, let $\bar{z} \in F^n$, and write $\ker$ for the kernel of exp.

$$\operatorname{td}(\mathbb{Q}(\exp(\bar{z}), \bar{\lambda})/\mathbb{Q}(\bar{\lambda})) + \operatorname{ldim}_{\mathbb{Q}(\bar{\lambda})}(\bar{z}/\ker) - \operatorname{ldim}_{\mathbb{Q}}(\bar{z}/\ker) \geqslant 0$$

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Outline of the proof in the real case.

Step 1

Take $\lambda \in \mathbb{R}$ exponentially transcendental, $B \cup \{\lambda\}$ an ecl-basis for $\mathbb{R}$, $C = \mathrm{ecl}(B)$. Then for any $\bar{z} \in \mathbb{R}^n$,

$$\mathrm{td}(\bar{z}, \lambda, \exp(\bar{z}), \exp(\lambda)/C) - \mathrm{ldim}_{\mathbb{Q}}(\bar{z}, \lambda/C) \geqslant 1$$

1. For each $a \in \mathbb{R}$, there is a C-definable function $\theta : \mathbb{R} \to \mathbb{R}$ with $\theta(\lambda) = a$.
2. If θ, ψ are two such, o-minimality implies $\{x \in \mathbb{R} \mid \theta(x) = \psi(x)\}$ contains an interval around λ.
3. Similarly, θ is differentiable near λ.
4. Define $\partial : \mathbb{R} \to \mathbb{R}$ by $a \mapsto \frac{d\theta}{dx}(\lambda)$, where $\theta(\lambda) = a$.
5. ∂ is a well-defined derivation on $\mathbb{R}$, vanishing on C.
6. $\partial e^{z_i} = e^{z_i}\partial z_i$, each i, and $\partial e^{\lambda} = e^{\lambda}\partial\lambda$.
7. Result follows from Ax's differential field version of Schanuel's Conjecture.

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7. Result follows from Ax's differential field version of Schanuel's Conjecture. □

Outline of the proof in the real case.

Step 1

Take $\lambda \in \mathbb{R}$ exponentially transcendental, $B \cup \{\lambda\}$ an ecl-basis for $\mathbb{R}$, $C = \text{ecl}(B)$. Then for any $\bar{z} \in \mathbb{R}^n$,

$$\text{td}(\bar{z}, \lambda, \exp(\bar{z}), \exp(\lambda)/C) - \text{ldim}_{\mathbb{Q}}(\bar{z}, \lambda/C) \geqslant 1$$

1. For each $a \in \mathbb{R}$, there is a C-definable function $\theta : \mathbb{R} \to \mathbb{R}$ with $\theta(\lambda) = a$.
2. If θ, ψ are two such, o-minimality implies $\{x \in \mathbb{R} \mid \theta(x) = \psi(x)\}$ contains an interval around λ.
3. Similarly, θ is differentiable near λ.
4. Define $\partial : \mathbb{R} \to \mathbb{R}$ by $a \mapsto \frac{d\theta}{dx}(\lambda)$, where $\theta(\lambda) = a$.
5. ∂ is a well-defined derivation on $\mathbb{R}$, vanishing on C.
6. $\partial e^{z_i} = e^{z_i}\partial z_i$, each i, and $\partial e^{\lambda} = e^{\lambda}\partial\lambda$.
7. Result follows from Ax's differential field version of Schanuel's Conjecture.

Step 2

$$\mathrm{td}(\exp(\bar{z})/\lambda) + \mathrm{ldim}_{\mathbb{Q}(\lambda)}(\bar{z}) - \mathrm{ldim}_{\mathbb{Q}}(\bar{z}) \geqslant 0$$

Proof

We have:

$$\begin{aligned}
1 &\leqslant \mathrm{td}(\bar{z}, \lambda, \exp(\bar{z}), \exp(\lambda)/C) - \mathrm{ldim}_{\mathbb{Q}}(\bar{z}, \lambda/C) \\
&= \mathrm{td}(\lambda/C) + \mathrm{td}(\bar{z}/C, \lambda) + \mathrm{td}(\exp(\bar{z})/C, \lambda, \bar{z}) \\
&\quad + \mathrm{td}(\exp(\lambda)/C, \lambda, \bar{z}, \exp(\bar{z})) - \mathrm{ldim}_{\mathbb{Q}}(\lambda/C, \bar{z}) - \mathrm{ldim}_{\mathbb{Q}}(\bar{z}/C) \\
0 &\leqslant \mathrm{td}(\bar{z}/C, \lambda) + \mathrm{td}(\exp(\bar{z})/\lambda) + \mathrm{td}(\exp(\lambda)/C, \exp(\bar{z})) \\
&\quad - \mathrm{ldim}_{\mathbb{Q}}(\lambda/C, \bar{z}) - \mathrm{ldim}_{\mathbb{Q}}(\bar{z}/C)
\end{aligned}$$

Also $\quad \mathrm{td}(\exp(\lambda)/C, \exp(\bar{z})) \leqslant \mathrm{ldim}_{\mathbb{Q}}(\lambda/C, \bar{z})$

and $\quad \mathrm{td}(\bar{z}/C, \lambda) \leqslant \mathrm{ldim}_{\mathbb{Q}(\lambda)}(\bar{z}/C)$

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Proof continued

Putting these together we get

$$\mathrm{td}(\exp(\bar{z})/\lambda) + \mathrm{ldim}_{\mathbb{Q}(\lambda)}(\bar{z}/C) - \mathrm{ldim}_{\mathbb{Q}}(\bar{z}/C) \geqslant 0$$

But $\mathbb{Q}(\lambda)$ is linearly disjoint from C over $\mathbb{Q}$, so

$$\mathrm{ldim}_{\mathbb{Q}(\lambda)}(\bar{z}/C) - \mathrm{ldim}_{\mathbb{Q}}(\bar{z}/C) \leqslant \mathrm{ldim}_{\mathbb{Q}(\lambda)}(\bar{z}) - \mathrm{ldim}_{\mathbb{Q}}(\bar{z})$$

which gives the result. □

Step 3

A similar argument shows for any $\bar{x}$:

$$\mathrm{ldim}_{\mathbb{Q}}(\lambda\bar{x}/\bar{x}) \geqslant \mathrm{ldim}_{\mathbb{Q}(\lambda)}(\bar{x})$$

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Theorem

For $\bar{y} \in \mathbb{R}^n_{>0}$ multiplicatively independent,

$$\operatorname{td}(\bar{y}, \bar{y}^\lambda / \lambda) \geqslant n$$

Proof.

- Step 2: $\forall \bar{z} \quad \operatorname{td}(\exp(\bar{z})/\lambda) + \operatorname{ldim}_{\mathbb{Q}(\lambda)}(\bar{z}) - \operatorname{ldim}_{\mathbb{Q}}(\bar{z}) \geqslant 0$
- Take $\bar{x} = \log \bar{y} \qquad \bar{z} = (\bar{x}, \lambda\bar{x})$
- Then

$$\begin{aligned} \operatorname{td}(\bar{y}, \bar{y}^\lambda/\lambda) &\geqslant \operatorname{ldim}_{\mathbb{Q}}(\bar{x}, \lambda\bar{x}) - \operatorname{ldim}_{\mathbb{Q}(\lambda)}(\bar{x}, \lambda\bar{x}) \\ &\geqslant \operatorname{ldim}_{\mathbb{Q}}(\bar{x}) + \operatorname{ldim}_{\mathbb{Q}}(\lambda\bar{x}/\bar{x}) - \operatorname{ldim}_{\mathbb{Q}(\lambda)}(\bar{x}) \\ &\geqslant n \end{aligned}$$

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