Some possible exponentiations over the universal enveloping algebra of $\mathbf{sl}_2(\mathbb{C})$

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Seminar's aim

We want to illustrate the main results of the work:

> *Some possible exponentiations over the universal enveloping algebra of $sl_2(\mathbb{C})$*
> *(S.L'I., A. Macintyre, F. Point).*

where some methods from model theory of modules and some techniques of ultraproducts are applied.

Outline

1. **Our Setting**
   Some results in this framework

2. **Exponentiation over $U = U_{\mathbb{C}}$**
   Exponential maps and ultraproducts

Outline

Our setting

Let k be an algebraically closed field of characteristic 0.

Consider the simple **Lie algebra** $sl_2(k)$ of

all 2×2 traceless matrices over k

with the bracket operation $[x, y] = xy - yx$.

Recall that a basis of $sl_2(k)$ is

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

So, $[x, y] = h$, $[h, x] = 2x$, $[h, y] = -2y$.

We focus on the universal enveloping algebra of $sl_2(k)$, denoted by U_k.

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We focus on the universal enveloping algebra of $sl_2(k)$, denoted by U_k.

Definition

A **universal enveloping algebra** of $sl_2(k)$ over k is

an associative algebra (with a unit) U_k with

a (Lie algebra) homomorphism $i : sl_2(k) \to U_k$ such that

if A is any associative k-algebra with the homomorphism

$$f : sl_2(k) \to A,$$

then there exists a unique homomorphism:

$$\Theta : U_k \to A$$

such that the diagram

$$\begin{array}{ccc} sl_2(k) & \to & U_k \\ \downarrow & \swarrow & \\ A & & \end{array}$$

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The Poincaré-Birkhoff-Witt Theorem

The k-algebra U_k has as basis (over k)

$$\{x^n y^l h^s \,:\, n, l, s \geq 0\}$$

where $\{x, y, h\}$ is the basis of $sl_2(k)$.

We will use these algebraic properties of U_k:

- U_k has a $\mathbb{Z}$-graded k-algebra. Let $U_{\kappa,m}$ be the subalgebra of elements of grade m. We have

$$\begin{aligned} U_k &= \bigoplus_{m\in\mathbb{Z}} U_{k,\,m}\,; \\ \text{for } m>0,\ U_{k,\,m} &= x^m U_{k,\,0} = U_{k,\,0}x^m\,; \\ \text{for } m<0,\ U_{k,\,m} &= y^{|m|} U_{k,\,0} = U_{k,\,0}y^{|m|}\,. \end{aligned}$$

- A key role is played by the **Casimir operator** of U_k:

$$c = 2xy + 2yx + h^2$$

which generates the center of U_k

- By PBW basis of U_k, we can see that the 0-component of U_k

$$U_{k\,0} = k[c,h]$$

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Simple finite dim. representations

Let λ be a positive integer.
Consider the vector space $k[X, Y]$.

Any simple $(\lambda + 1)$-dim. $sl_2(k)$-module V_λ can be described as the subspace of $k[X, Y]$
of all homogenous polynomials in X and Y of degree λ.

According to the following basis of monomials

$$X^\lambda, X^{\lambda-1}Y, \ldots, XY^{\lambda-1}, Y^\lambda,$$

we have

$$V_\lambda = \bigoplus_{j=0}^{\lambda} kX^{\lambda-j}Y^j.$$

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$$V_\lambda = \bigoplus_{j=0}^{\lambda} kX^{\lambda-j}Y^j\,.$$

A representation of $sl_2(k)$ is given by the map $f_\lambda : sl_2(k) \rightarrow End(V_\lambda)$ defined as follows:

$$f_\lambda(x) = X\frac{\partial}{\partial Y}$$
$$f_\lambda(y) = Y\frac{\partial}{\partial X},$$
$$f_\lambda(h) = X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}.$$

A classification by I.Herzog

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[Herzog]

The pseudo-finite dimensional representations of $\mathrm{sl}(2, k)$.

Selecta Mathematica 7 (2001), 241-290

A classification by I.Herzog

On the language of left U_k-modules, a classification of simple representations of U_k is given by I.Herzog.

1. Let U'_k be the ring of definable scalars of all simple finite dimensional U_k-modules whose elements are pp-definable endomorphisms of each V_λ.

- Herzog proved that U'_k is von Neuman regular ring.

A classification by I.Herzog

On the language of left U_k-modules, a classification of simple representations of U_k can be given by I.Herzog.

2. A representation M of U_k is called **pseudo-finite dimensional** (**PFD**) iff
   M satisfies all sentences (of the language of U_k-modules) true in every finite dimensional representation.

- He investigated these representations, viewed as modules over U'_k, by analyzing the Ziegler spectrum of U'_k.

Some works inspired by Herzog's analysis

[L'I., Prest]

Rings of definable scalars of Verma modules, 2007

[Herzog, L'I.]

The nonstandard quantum plane, 2008

[L'I., Macintyre]

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Exponentiation

Restrict our attention on $\mathbb{C}$. Let $U = U_{\mathbb{C}}$.

Our aim We define some possible exponentiations over U.

1. First, we describe the exponential map

$$\mathrm{EXP}_\lambda : U \longrightarrow GL_{\lambda+1}(\mathbb{C})$$

for each $\lambda \in \omega - \{0\}$.

2. Then, we discuss the exponential map

$$\mathrm{EXP} : U \to \prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C})$$

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Our strategy

We will use:

- The *matrix characterization* of every simple U-modules V_λ by the map $\Theta_\lambda : U \to M_{\lambda+1}$ (where $M_{\lambda+1} = \mathrm{End}(V_\lambda)$).
- The natural matrix exponential map defined over $M_{\lambda+1}(\mathbb{C})$

$$\exp : M_{\lambda+1}(\mathbb{C}) \longrightarrow GL_{\lambda+1}(\mathbb{C})$$

such that $\forall A \in M_{\lambda+1}(\mathbb{C})$,

$$exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I_{\lambda+1} + A + \frac{A^2}{2} + \frac{A^3}{3!} + \ldots)$$

where $I_{\lambda+1}$ denote the $(\lambda+1) \times (\lambda+1)$ identity matrix.

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Definition: the map EXP_λ

Let $\lambda \in \omega - \{0\}$ (later λ will range in ω).
We can define a **new exponential map** over U:

$$\text{EXP}_\lambda : U \xrightarrow{\Theta_\lambda} M_{\lambda+1}(\mathbb{C}) \xrightarrow{\exp} GL_{\lambda+1}(\mathbb{C})$$

$$\text{EXP}_\lambda(u) = \exp(\Theta_\lambda(u)), \qquad \forall u \in U.$$

Proposition

We can prove that the map EXP_λ is surjective.

Question.

Which is the value of $\text{EXP}_\lambda(u)$ for every $u \in U$? What is its kernel?

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But, we can easily calculate EXP_λ of x, y, h, c by the related values of Θ_λ:

$$\Theta_\lambda(x) = \begin{pmatrix} 0 & 1 & 0\dots & 0 \\ 0 & 0 & 2\dots & 0 \\ \vdots & \vdots & & \lambda \\ 0 & 0 & 0\dots & 0 \end{pmatrix}, \Theta_\lambda(y) = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \lambda & 0 & \dots & 0 \\ 0 & \lambda-1 & & 0 \\ \vdots & \vdots & & \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\Theta_\lambda(h) = \mathrm{diag}\,(\lambda, \lambda-2, \dots, -\lambda+2, -\lambda)\,.$$

Since Θ_λ is a homomorphism, we can easily calculate

$$\begin{aligned} \Theta_\lambda(c) &= \Theta_\lambda(2x\cdot y + 2y\cdot x + h^2) = \\ &= 2\Theta_\lambda(x)\cdot\Theta_\lambda(y) + 2\Theta_\lambda(y)\cdot\Theta_\lambda(x) + (\Theta_\lambda(h))^2 = \\ &= \mathrm{diag}\left(\lambda^2+2\lambda, \dots, \lambda^2+2\lambda\right). \end{aligned}$$

Because of the intrinsic characterization of U, we are not able to give immediately a satisfactory answer.
But, we can easily calculate EXP_λ of x, y, h, c by the related values of Θ_λ:

$$\begin{aligned}
\mathrm{EXP}_\lambda(x) &= \exp(\Theta_\lambda(x)) = \\
&= 1_{\lambda+1} + \Theta_\lambda(x) + \frac{\Theta_\lambda(x)^2}{2} + \ldots + \frac{\Theta_\lambda(x)^\lambda}{\lambda!};
\end{aligned}$$

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\mathrm{EXP}_\lambda(y) &= \exp(\Theta_\lambda(y)) = \\
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$$\begin{aligned}
\mathrm{EXP}_\lambda(h) &= \exp(\Theta_\lambda(h)) = \\
&= \mathrm{diag}(e^\lambda, e^{\lambda-2}, \ldots, e^{-\lambda+2}, e^{-\lambda}); \\
\mathrm{EXP}_\lambda(c) &= \exp(\Theta_\lambda(c)) = \mathrm{diag}(e^{\lambda^2+2\lambda}, \ldots, e^{\lambda^2+2\lambda})
\end{aligned}$$

We can prove that EXP_λ satisfies the similar properties of the matrix exponential exp.

Proposition

If $u, v \in U$:

(i) $\mathrm{EXP}_\lambda(0_U) = I_{\lambda+1}$, where 0_U denotes the identity element (with respect to the addition) in U;

(ii) $\mathrm{EXP}_\lambda(u) \cdot \mathrm{EXP}_\lambda(-u) = I_\lambda$;

(iii) for u and v commuting, $\mathrm{EXP}_\lambda(u+u) = \mathrm{EXP}_\lambda(u) \cdot \mathrm{EXP}_\lambda(v)$;

(iv) for an invertible element v in U, $\mathrm{EXP}_\lambda(vuv^{-1}) = \Theta_\lambda(v)\mathrm{EXP}_\lambda(u)\Theta_\lambda(v)^{-1}$;

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Remark

Any element $u_0 \in U_0$ belongs to the kernel of EXP_λ if and only if

$$\bigwedge_{0 \leq j \leq \lambda} p\left(\lambda^2 + 2\lambda, \lambda - 2j\right) \in 2\pi i \mathbb{Z}$$

We can get a partial answer to our question.

Proposition

EXP_λ maps any element u of U onto $SL_{\lambda+1}(\mathbb{C})$ if the following condition is satisfied

$$\mathrm{tr}(\Theta_\lambda(u)) \in 2\pi i \mathbb{Z}.$$

In particular, if $u \in \oplus_{m \neq 0} U_m$, then its image by EXP_λ lies always in $SL_{\lambda+1}(\mathbb{C})$.

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A further aim

Let $\mathcal{V}$ be a non-principal ultrafilter on ω and consider the ultraproducts $\prod_{\mathcal{V}} M_{\lambda+1}(\mathbb{C})$ and $\prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C})$ as structures on the language of Lie algebras.

We will focus on the map EXP from U to $\prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C})$ defined as follows:

$$\mathrm{EXP} : U \to \prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C})$$

$$\mathrm{u} \to [EXP_{\lambda}(\mathrm{u})]_{\mathcal{V}} \qquad \forall u \in U$$

by composing the injective map $[\Theta_{\lambda}] : U \to \prod_{\mathcal{V}} M_{\lambda+1}(\mathbb{C})$, with the map $[\exp]_{\mathcal{V}} : \prod_{\mathcal{V}} M_{\lambda+1}(\mathbb{C}) \to \prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C})$.

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Note that *EXP* satisfies the properties stated for each EXP_λ. Moreover,

- $EXP(\oplus_{m \neq 0} U_m) \subset \prod_{\mathcal{V}} SL_{\lambda+1}(\mathbb{C})$;
- $EXP(U_0) \subset \prod_{\mathcal{V}} Diag_{\lambda+1}(\mathbb{C})$.

We focus on the following query.

Question

What is the kernel of EXP?

Proposition

Let $u := p(c, h) \in U_0$, where $p[x_1, x_2] \in \mathbb{C}[x_1, x_2]$ is in the form $\frac{1}{2\pi \cdot i} \cdot q[x_1, x_2]$. Write $q(x_1, x_2) = \sum_{k=0}^{d} q_k(x_1) x_2^k$, with $q_k(x) \in \mathbb{Q}[x_1]$.

Then, $p \in \mathrm{Ker}(\mathrm{EXP})$ for all non-principal ultrafilter $\mathcal{V}$ if and only if $q(x_1, x_2) \in \mathbb{Q}[x_1, x_2]$ and for each $0 \leq k \leq d$, $q_k(0) \in \mathbb{Z}$.

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Further questions

We would like to put a topology on U in such a way that *EXP* is continuous.

The sesquilinear Hermitian forms $(\cdot,\cdot)_\lambda$ induce on the Lie algebra $\prod_\mathcal{V} M_{\lambda+1}(\mathbb{C})$ (over $\mathbb{C}^* = \prod_\mathcal{V} \mathbb{C}$) a $\star$-Hermitian sesquilinear form $(\cdot,\cdot)$ defined by:

$$([A_\lambda]_\mathcal{V}, [B_\lambda]_\mathcal{V}) := [(A_\lambda, B_\lambda)]_\mathcal{V}.$$

So, we have a $\star$-norm $\|\cdot\|$ on $\prod_\mathcal{V} M_{\lambda+1}(C)$,

$$\|[A_{\lambda+1}]\| := [\|A_{\lambda+1}\|_{\lambda+1}].$$

which induces on U the following $\star$-norm (also denoted by $\|\cdot\|$):

$$\|u\| := [\|\Theta_\lambda(u)\|_{\lambda+1}]$$

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Proposition

Consider the $\star$-normed spaces $(U, \|\cdot\|)$ and $(\prod_{\mathcal{V}} M_{\lambda+1}(\mathbb{C}), \|\cdot\|_{\lambda+1})$. The map $EXP : U \to \prod_{\mathcal{V}} GL_{\lambda+1}(\mathbb{C})$ is continuous and maps bounded sets to bounded sets.

Proof

Let $\epsilon \in \prod_{\mathcal{V}} \mathbb{R}^{>0}$, let $\eta := 2^{-1} \cdot \epsilon \cdot e^{-\|u\|}$, and let $v \in O_\eta$. Then $\|EXP(u+v) - EXP(u)\| \leq \eta e^{\|u\|}.e^{\eta}$.

If the sequence $A_{\lambda+1} \in M_{\lambda+1}(\mathbb{C})$ is bounded, then the corresponding sequence $\|\exp(A_{\lambda+1})\|_{\lambda+1}$ is bounded.

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We can extend the exponential map EXP to $U \otimes \mathbb{R}^*$ (where $\mathbb{R}^* = \prod_{\mathcal{V}} \mathbb{R}$.

A topological group G is $\star$-path connected if $\forall h_0, h_1 \in G$, $\exists$ a continuous map $g : [0;1]^* \to G$ (where $[0;1]^* := \mathbb{R}^* \cap [0;1]$) such that $g(0) = h_0$ and $g(1) = h_1$.

Proposition

The subgroups $< \mathrm{EXP}(U) >$ and $\mathrm{EXP}(U_0)$ (respectively $< \mathrm{EXP}(U \otimes \mathbb{R}^*) >$ and $\mathrm{EXP}(U_0 \otimes \mathbb{R}^*)$ are topological groups.

Moreover, $< \mathrm{EXP}(U \otimes \mathbb{R}^*) >$ and $\mathrm{EXP}(U_0 \otimes \mathbb{R}^*)$ are $\star$-path connected.

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A topological group G is $\star$-path connected if $\forall h_0, h_1 \in G$, $\exists$ a continuous map $g : [0;1]^* \to G$ (where $[0;1]^* := \mathbb{R}^* \cap [0;1]$) such that $g(0) = h_0$ and $g(1) = h_1$.

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The subgroups $< \mathrm{EXP}(U) >$ and $\mathrm{EXP}(U_0)$ (respectively $< \mathrm{EXP}(U \otimes \mathbb{R}^*) >$ and $\mathrm{EXP}(U_0 \otimes \mathbb{R}^*)$ are topological groups.

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The asymptotic cone

Define the map $\phi : M_{\lambda+1}(\mathbb{C}) \to \omega$ which sends every $A \in M_{\lambda+1}(\mathbb{C})$ to the number of non-zero coefficients of A. Let us check that

1. $\phi(A+B) \leq \phi(A) + \phi(B)$,
2. $\phi(A \cdot B) \leq \phi(A) \cdot \phi(B)$

ϕ defines a norm on $M_{\lambda+1}(\mathbb{C})$, denoted by $\| \cdot \|_{c,\lambda+1}$.

Let $\prod^*_{\mathcal{V}}(M_{\lambda+1}(\mathbb{C}), \frac{\|\cdot\|_{c,\lambda+1}}{\lambda})$ be the set of elements $[a_\lambda] \in \prod_{\mathcal{V}}(M_{\lambda+1}(\mathbb{C}), \frac{\|\cdot\|_{c,\lambda+1}}{\lambda})$ such that for $N \in \omega$,

$$\{\lambda \in \omega : \; \|a_\lambda\|_{c,\lambda} \leq N \cdot \lambda\} \in \mathcal{V}.$$

The asymptotic cone

Let $X_{\mathcal{V}} := \prod_{\mathcal{V}}^{*}(M_{\lambda+1}(\mathbb{C}), \frac{\|\cdot\|_{c,\lambda+1}}{\lambda})/\sim$, where the equivalence relation $\sim$ is defined by

$$[a_\lambda]_{\mathcal{V}} \sim [b_\lambda]_{\mathcal{V}} \text{ if } \frac{\|a_\lambda - b_\lambda\|_{c,\lambda}}{\lambda} \to_{\mathcal{V}} 0.$$

$X_{\mathcal{V}}$ becomes a metric space $(X_{\mathcal{V}_\lambda}(\mathbb{C}), d)$ with the distance

$$d(a,b) := st\left(\left[\frac{\|a_\lambda - b_\lambda\|_{c,\lambda}}{\lambda}\right]\right) \quad \forall a, b \in X_{\mathcal{V}}$$

where st denote the standard part of an element of $\mathbb{R}^*$ whose absolute value is bounded by some natural number.

Proposition

U embeds in $(X_{\mathcal{V}}(\mathbb{C}), d)$.

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