Superstable groups acting on trees

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- A group G acts *freely* if every nontrivial element of G is hyperbolic.

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Remark. A superstable group acting freely on a real (or simplicial) tree is abelian.

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What can be said about the model theory of groups acting nontrivially on simplicial trees? Is it possible for such groups to be ω-stable or superstable?

Bass-Serre theorem

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A group acts without inversions and nontrivially on a simplicial tree if and only if either G splits as a free product with amalgamation or G has an infinite cyclic quotient. □

Free products

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A nontrivial free product $G_1 * G_2$ *is superstable if and only if* $G_1 = G_2 = \mathbb{Z}_2$. □

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Using Bass-Serre theorem:

Corollary 1

A free product with amalgamtion or an HNN-extension is not ω-stable. □

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- If G is superstable then $G \oplus (\mathbb{Z}_2 * \mathbb{Z}_2)$ is superstable and acts nontrivially on a simplicial tree. Moreover $G \oplus (\mathbb{Z}_2 * \mathbb{Z}_2) = (G \oplus \mathbb{Z}_2) *_G (G \oplus \mathbb{Z}_2)$.

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- (Fact) g is hyperbolic if and only if $\ell(g) > 0$.

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(2)(Dihedral actions) the hyperbolic length function is given by $\ell(g) = |\rho(g)|$ for $g \in G$, where $\rho : G \to \mathrm{Isom}(\Lambda)$ is a homomorphism whose image contains a reflection and a nontrivial translation, and the absolute value signs denote hyperbolic length for the action of $\mathrm{Isom}(\Lambda)$.

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(3)(Irreducible actions) G contains a free subgroup of rank 2 which acts freely, without inversions and properly discontinuously on T.

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Corollary 2

If G is superstable and splits as $G = G_1 *_A G_2$, with the index of A in G_1 different from 2, then G interprets a simple superstable non ω-stable group acting nontrivially on a simplicial tree. □

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(1) *H_1 is connected, any action of H_1 on a Λ-tree is trivial, H_2/H_1 is soluble and has a nontrivial action on a Λ-tree.*

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(2) *H_2/H_1 is simple and acts nontrivially on a Λ-tree, H_2/H_1 has a Borel family of equationally-definable nilpotent subgoups such that there exists $m \in \mathbb{N}$ such that for every hyperbolic element g in H_2/H_1, there is $1 \leq n \leq m$, such that g^n is in some $B \in \mathcal{B}$.*

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