Semiabelian varieties over function fields, logarithmic derivatives, and exponentiation (Joint work with D. Bertrand)

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Aims of the talk

- I will talk about a functional/differential algebraic analogue of Lindemann's theorem, for semiabelian varieties G over function fields K, whose statement is still moving.
- Lindemann's theorem says that if $x_1, .., x_n$ are $\mathbb{Q}$-linearly independent algebraic numbers, then $e^{x_1}, .., e^{x_n}$ are algebraically independent. It is the "exponential side" of Schanuel's conjecture that $tr.deg(\mathbb{Q}(x_1, .., x_n, e^{x_1}, .., e^{x_n})/\mathbb{Q}) \geq n$ for an arbitrary set $(x_i)_i$ of $\mathbb{Q}$-linearly independent complex numbers.
- The novelty, compared with say work of Ax on the function field case, is that we will allow "nonconstant" semiabelian varieties.
- I will always concentrate on the "exponential" side where the x_i's are rational over the base field K, even though some methods give information on other cases such as the logarithmic side too.

The functional case for algebraic tori I

- Let K be an algebraically closed field of transcendence degree 1 over $\mathbb{C}$. We can equip K with a derivation ∂ with field of constants $\mathbb{C}$ (e.g ∂ extends d/dt.)
- If $x \in K$, $y = exp(x)$ makes sense, as a point in a larger differential field F: $x \in K_0$ for some finitely generated differential subfield of K containing $\mathbb{C}$. So x can be viewed as a rational function on a complex curve S, so $exp(x)$ lives in a differential field F_0 of meromorphic functions on some small disc in S, and can be jointly embedded with K over K_0 into suitable F.
- Moreover the differential relation $\partial y/y = \partial x$ is satisfied by any (y, x) for which $y = exp(x)$.

Theorem 1.1

(Exponential side of Ax) Suppose $x_1, .., x_n \in K$ *are* $\mathbb{Q}$*-linearly independent modulo* $\mathbb{C}$*. Then*
(i) if $y_1, .., y_n$ *are elements of a differential field* $F > K$ *such that* $\partial y_i / y_i = \partial x_i$ *for* $i = 1, ., n$ *then* $y_1, .., y_n$ *are algebraically independent over* K*.*
(ii) In particular if $y_i = exp(x_i)$ *for* $i = 1, , ., n$ *then* $y_1, .., y_n$ *are algebraically independent over* K*.*

Note that in this functional setting, the "modulo $\mathbb{C}$" part of the hypothesis is needed.

Proof. (i)

- If not then we may choose such solutions $y_1, .., y_n$ in K^{diff} with $tr.deg(K(y_1, .., y_n)/K) < n$.
- Let $a_i = \partial x_i \in K$. So $(y_1, .., y_n)$ is a solution of the system $\partial y_i = a_i y_i$, $i = 1, .., n$ of linear differential equations.
- $L = K(y_1, .., y_n)$ is a Picard-Vessiot extension of K.
- In fact if $\sigma \in Aut(L/K)$ then $\sigma(y_i) = y_i \cdot b_i(\sigma)$ for some unique $b_i(\sigma) \in \mathbb{C}^*$, and the map which takes σ to $(b_1(\sigma), .., b_n(\sigma))$ is an isomorphism of $Aut(L/K)$ with a proper algebraic subgroup H of ${\mathbb{C}^*}^n$.
- H is defined by equations $z_1^{k_1} \cdot .. \cdot z_n^{k_n} = 1$ ($k_i \in \mathbb{Z}$, not all 0).
- Hence for some such $k_1, .., k_n$ we have that $b_1(\sigma)^{k_1} \cdot ... \cdot b_n(\sigma)^{k_n} = 1$ for all $\sigma \in Aut(L/K)$.

The functional case for algebraic tori IV

- Then check that $\sigma(y) = y$ for all $\sigma \in Aut(L/K)$, where $y = y_1^{k_1} \cdot .. \cdot y_n^{k_n}$.
- But then $y \in K$.
- It is clear that $\partial y/y = \partial x$ where $x = k_1 x_1 + .. + k_n x_n$, and $x \notin \mathbb{C}$ by hypothesis.
- So we have reduced the theorem to the case $n = 1$, which states essentially that a rational function $f(z)$ cannot be both a derivative and a logarithmic derivative, unless it is 0, And this is left to the reader.

End of proof.

The functional case for arbitrary semiabelian varieties over $\mathbb{C}$ I

- For G a commutative connected n-dimensional algebraic group over $\mathbb{C}$ and $LG = \mathbb{G}_a^n$ its Lie algebra, we have $exp_G : LG(\mathbb{C}) = \mathbb{C}^n \to G(\mathbb{C})$, an analytic surjective homomorphism between the two complex Lie groups, characterized by its differential at 0 being the identity.
- We have Kolchin's logarithmic derivative $\partial \ell n_G : G \to LG$. This is a first order differential rational homomorphism, surjective when considering points in a differentially closed field, and with kernel the constants in whichever differential field the map is being evaluated.
- For example if G is an elliptic curve over $\mathbb{C}$ in standard form $\partial \ell n_G$ is $\partial x / y$.
- We just write $\partial : \mathbb{G}_a^n \to \mathbb{G}_a^n$ for the map taking $(x_1, .., x_n)$ to $(\partial(x_1), .., \partial(x_n))$.

The functional case for arbitrary semiabelian varieties over $\mathbb{C}$ II

- If K is as before (tr.deg 1 algebraically closed extension of $\mathbb{C}$ with derivation ∂), and $x \in LG(K) = K^n$, then $y = exp_G(x) \in G(F)$ for suitable $F > K$ makes sense, and we have:
- $\partial \ell n_G(y) = \partial(x)$
- We consider a semiabelian variety G defined over $\mathbb{C}$, namely we have an exact sequence $T \to G \to A$ of comnmutative algebraic groups over $\mathbb{C}$ with T an algebraic torus and A an abelian variety.
- Let $\tilde{G}$ be the "universal vectorial extension" of G. Namely $\tilde{G}$ is an extension of G by some vector group $W = \mathbb{G}_a^m$ and for any other such extension H of G there unique $\tilde{G} \to H$ with everything commuting.

The functional case for arbitrary semiabelian varieties over $\mathbb{C}$ III

Theorem 1.2

(Exponential side of Ax-Kirby-Bertrand) Let G be a semiabelian variety over $\mathbb{C}$, and let $x \in LG(K)$ be such that $x \notin LH(K) + LG(\mathbb{C})$ for any proper algebraic subgroup H of G.
(i) Let y be any solution of $\partial \ell n(y) = \partial(x)$ in a differential field F extending K. Then $tr.deg(K(y)/K) = dim(G)$. In particular $tr.deg(K(exp_G(x))/K) = dim(G)$.
(ii) Let $\tilde{x} \in L\tilde{G}(K)$ be any lift of x. Then again for any solution $\tilde{y}$ of $\partial \ell n(-) = \partial(\tilde{x})$ we have that $tr.deg(K(\tilde{y})/K) = dim(\tilde{G})$. In particular $tr.deg(K(exp_{\tilde{G}}(\tilde{x}))/K) = dim(\tilde{G})$.

Again this result reduces, via differential Galois theory, to showing that $y \notin G(K)$ in some "irreducible" contexts.

Nonconstant case - background I

- Let K be as before and we will consider commutative connected algebraic groups G defined over K.
- We call G constant if G is isomorphic as an algebraic group to one defined over $\mathbb{C}$.
- G always has a maximal constant algebraic subgroup, denoted by $G_{(0)}$.
- There are at least two sources of nonconstant G; first nonconstant abelian varieties, such as the elliptic curve $y^2 = x(x-1)(x-t)$ where $t \in K \setminus \mathbb{C}$.
- Secondly nonconstant extensions of a constant abelian varieties A by an algebraic torus: the extensions of A by $\mathbb{G}_m$ have a moduli space (which is the dual abelian variety $\hat{A}$).

Nonconstant case - background II

- If A is an abelian variety over K then up to isogeny $A = A_0 \times A_1$ where A_0 is constant, and A_1 of $\mathbb{C}$-trace 0 (totally nonconstant).
- If $T \to G \to A$ is a semiabelian variety, let G_0 denote the preimage in G of A_0 and call it the *semiconstant* part of G. So $G_{(0)} \subseteq G_0$.

Nonconstant case - exp

- For G a commutative connected algebraic group over K and $LG = \mathbb{G}_a^n$ its Lie algebra, and for $x \in LG(K)$ we can speak of $exp_G(x)$, as a point in a larger differential field:
- Again $x \in LG(K_0) = \mathbb{C}(S)$ for some complex curve S with all data defined over K_0.
- G is the "generic fibre" of a fibration $\mathbf{G} \to S$ of complex varieties, where the fibres $\mathbf{G}_s$ are complex algebraic groups.
- Likewise there is a corresponding complex vector bundle $\mathbf{LG} \to S$ whose generic fibre is LG.
- $x \in LG(K_0)$ is then a rational *section* of $\mathbf{LG} \to S$, holomorphic on some small S_0.
- Applying appropriate exp's in the fibres, gives us a holomorphic section $exp_{\mathbf{G}}(x)$ of $\mathbf{G} \to S$ above S_0, which we call $exp_G(x)$, and lives in the differential field of meromorphic functions on S_0, which extends K_0.

Nonconstant case - logarithmic derivatives I

- Let now G be a possibly nonconstant semiabelian variety over K
- To obtain an appropriate analogue of the differential relation $\partial \ell n(y) = \partial(x)$ which was satisfied by the graph of exponentiation in the constant case, we are in general *forced* to pass to the universal vectorial extension $\tilde{G}$ of G.
- The point is that $\tilde{G}$ has a (unique) so-called D-group structure, namely an extension ∂' of ∂ on K to a derivation of the "coordinate ring" of $\tilde{G}$ which respects co-multiplication.
- Equivalently, a K-rational homomorphic section $s : G \to T_\partial(\tilde{G})$, where the latter is the "first prolongation" or "shifted tangent bundle" of $\tilde{G}$.
- This yields our logarithmic derivative $\partial \ell n_{\tilde{G}} : \tilde{G} \to L\tilde{G}$:

Nonconstant case - logarithmic derivatives II

- For F a differential field extending K and $g \in \tilde{G}(K)$, $\partial\ell n_{\tilde{G}}(g) = \partial(g) - s(g)$ where $-$ is in the sense of the canonical group structure on $T_{\partial}\tilde{G}$. (Same definition works to give Kolchin's log.derivative in the constant case, taking $s = 0$.)
- The D-structure on $\tilde{G}$ gives rise to the "connection" $\partial_{L\tilde{G}}$ on $L\tilde{G}$:
- Either by differentiating (in the sense of Kolchin) $\partial\ell n_{\tilde{G}}$ at the identity, or by considering the map from the cotangent space of $\tilde{G}$ at the identity to itself, induced by the derivation ∂' (as in the work with Ziegler).
- In any case $\partial_{L\tilde{G}} : L\tilde{G} \to L\tilde{G}$ is additive and satisfies the Leibniz rule with respect to scalar multiplication, namely equips the vector space $L\tilde{G}$ with a ∂-module structure, but now possibly nontrivial.

Nonconstant case - logarithmic derivatives III

- When A is an abelian variety over K, then $L\tilde{A}$ identifies with the dual of the de Rham cohomology group $H^1_{dR}(A)$, and $\partial_{L\tilde{G}}$ coincides with the dual of the standard Gauss-Manin connection on $H^1_{dR}(A)$.
- In any case for $\tilde{x} \in L\tilde{G}(K)$, and $\tilde{y} = exp_{\tilde{G}}(\tilde{x})$ it is again the case that $\partial \ell n_{\tilde{G}}(\tilde{y}) = \partial_{L\tilde{G}}(\tilde{x})$, although with our differential algebraic definitions above, this requires some work to verify.
- We are now in a position to state the main theorem, of which Theorem 1.2 above is a special case.

Main theorem and remarks I

Theorem 2.1

Let G be a semiabelian variety over K. Let $x \in LG(K)$. Assume that

Hyp_x: $x \notin LH(K) + LG_{(0)}(\mathbb{C})$ for any proper algebraic subgroup H of G, and moreover for any algebraic subgroup G_1 of G, the same holds for the image of x in $L(G/G_1)$.

Let $\tilde{x} \in L\tilde{G}(K)$ be any lift of x. Then

(i) If $\tilde{y}$ is any solution of $\partial\ell n_{\tilde{G}}(-) = \partial_{L\tilde{G}}(\tilde{x})$ in a differential field $(F, \partial) \supseteq (K, \partial)$ then $tr.deg(K(\tilde{y})/K) = dim(\tilde{G})$.

(ii) In particular $tr.deg(K(exp_{\tilde{G}}(\tilde{x}))/K) = dim(\tilde{G})$, and so $tr.deg(K(exp_G(x)/K) = dim(G)$.

Main theorem and remarks II

- The hypothesis Hyp_x is easily seen to be necessary. But when the semiconstant part G_0 of G coincides with the constant part $G_{(0)}$, then the moreover clause in Hyp_x follows from the first clause, so can be dispensed with.
- But in the simplest case where the semiconstant part of G is not constant, namely when G is a nonconstant extension of a constant elliptic curve E by $\mathbb{G}_m$, the moreover clause canNOT be dropped. Even to see this counterexample requires results around variation of mixed Hodge structure.
- Note that when $G = A$ is an abelian variety with $\mathbb{C}$-trace 0 then Hyp_x says simply that $x \notin LB(K)$ for any proper abelian subvariety of A, and is a *direct* translation of the hypothesis on $x_1, .., x_n$ in the number theoretic situation (Theorems 1.1, 1.2).

Main theorem and remarks III

- Applying Theorem 2.1 to the case where G is a power of a nonconstant elliptic curve, one obtains:
- If $\wp$ is an elliptic function with nonconstant invariant $j \in \mathbb{C}(z)$ and zeta function ζ, and if $x_1(z), .., x_n(z)$ are $\mathbb{Z}$-linearly independent algebraic functions, then the $2n$ analytic functions defined on some open domain in $\mathbb{C}$ by $\wp(x_1(z)), .., \wp(x_n(z)), \zeta(x_1(z)), .., \zeta(x_n(z))$ are algebraically independent over $\mathbb{C}(z)$.

Comments on the proof I

- The proof of Theorem 2.1 is inductive in nature and takes us into the category of "almost semiabelian D-groups".
- Deligne's theorem of the fixed part (that the set of K-rational solutions of the linear DE $\partial_{L\tilde{A}}(-) = 0$ is trivial when A is abelian and traceless) plays a role.
- There are essentially two base cases of the inductive proof. The first can be taken to be the case when G is constant (so Theorem 1.2).
- The second is a kind of $n = 1$ case of the other extreme: and says that when $G = A$ is simple and of $\mathbb{C}$-trace 0, $x \in LA(K)$ is nonzero, and $\tilde{x} \in L\tilde{A}(K)$ is an arbitrary lift of x, then there is NO $\tilde{y} \in \tilde{A}(K)$ satisfying $\partial\ell n_{\tilde{A}}(\tilde{y}) = \partial_{L\tilde{A}}(\tilde{x})$.
- The latter is precisely Manin's "theorem of the kernel" in the form discussed by Coleman and proved by Chai.

Comments on the proof II

- We call $\tilde{G}$ K-large, if working in the differential closure K^{diff} of K, the kernel of $\partial\ell n_{\tilde{G}}$ is contained in $\tilde{G}(K)$.
- If $\tilde{G}$ is K-large, then the reduction to the two special cases above can be effected via (generalized) differential Galois theory, as in our proof of Theorem 1.1 above.
- However K-largeness of $\tilde{G}$ is a rather restrictive condition. But it holds for example if G is a product of a torus, a constant A_0 and a "general" traceless A_1.
- To effect the inductive proof in general we need the "socle theorem" (from the jet space paper): If G is a connected finite-dimensional differential algebraic group and X is an irreducible differential algebraic subvariety of G with trivial stabilizer, then X is contained in a coset of the maximal "split" or "algebraic" connected differential algebraic subgroup of G.

Comments on the proof III

- Even in this exponential side of nonconstant Ax, our statement is not optimal. One would like for example, for arbitrary $x \in LG(K)$ a geometric object attached to x which governs the relevant transcendence degrees (as in the usual statements of Ax).
- One would again look for such statements in the logarithmic and mixed cases, although some work on the logarithmic case already appears in Bertand's paper in the Newton volume.