Abelian categories and imaginaries

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More generally, make the same definitions but now with R replaced by any skeletally small preadditive category $\mathcal{R}$.

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ev_D is an exact functor;
$\mathcal{D} \simeq \mathrm{Ex}\big(\mathbb{L}(\mathcal{D})^{\mathrm{eq}+}, \mathbf{Ab}\big)$, the category of exact functors from $\mathbb{L}(\mathcal{D})^{\mathrm{eq}+}$ to $\mathbf{Ab}$.

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and if $\mathcal{D}$ is a definable subcategory of $\mathrm{Mod}\text{-}\mathcal{R}$ then $\mathbb{L}(\mathcal{D})^{\mathrm{eq}+}$ is the quotient category/localisation of $(\mathrm{mod}\text{-}R, \mathbf{Ab})^{\mathrm{fp}}$ by the Serre subcategory of those functors which are 0 on $\mathcal{D}$.

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$\mathbb{L}(\mathcal{D})^{\mathrm{eq}+}$ can be any small abelian category: given a skeletally small abelian category $\mathcal{A}$, set $\mathcal{D} = \mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$. Then $\mathcal{D}$ is a definable subcategory of $(\mathcal{A}, \mathbf{Ab}) = \mathcal{A}\text{-}\mathrm{Mod} = \mathrm{Mod}\text{-}\mathcal{A}^{\mathrm{op}}$ and $\mathcal{A} \simeq \mathbb{L}(\mathcal{D})^{\mathrm{eq}+}$.

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Let $\mathbb{ABEX}$ denote the 2-category whose objects are the skeletally small abelian categories, whose (1-)arrows are the exact functors and whose 2-arrows are the natural transformations; let $\mathbb{DEF}$ denote the 2-category whose objects are the definable additive categories, whose (1-)arrows are the functors which commute with direct products and direct limits (i.e. the interpretation functors) and whose 2-arrows are the natural transformations.

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The above gives an equivalence of 2-categories.