THE VALUATION INEQUALITY

VI states that

tr. deg (a valued field) $\geq$ rank(value group) + tr. deg (residue field).

- True for characteristic 0, 0.
- "tr. deg." may be taken over any subfield k_0 of the valuation ring (so $k_0 \subseteq$ residue field).

In this talk all fields will be (explicitly) subfields of $^*\mathbb{C} := \mathbb{C}^{\mathbb{N}}/\mathcal{U}$, with valuation v induced by the valuation ring

$$\mathrm{Fin} := \{a \in {}^*\mathbb{C} : \exists r \in \mathbb{R} \ (\subseteq {}^*\mathbb{C}), \ |a| \leq r\}.$$

So value group $\subseteq \langle {}^*\mathbb{R}/\mathrm{Fin}, +, < \rangle := \Gamma$

$$v : {}^*\mathbb{C} \setminus \{0\} \longrightarrow \Gamma : z \longmapsto -\log|z| / \mathrm{Fin}$$

(NB $\log|z+w| \leq \max\{\log|z|, \log|w|\} + \log 2$.)

- Will use VI to study asymptotic relations between functions defined using $+, \cdot, \exp$.

NB, let $\mu :=$ maximal ideal of Fin

$$= \{a \in {}^*\mathbb{C} : \forall r \in \mathbb{R}, r > 0 \Rightarrow |a| < r\}.$$

Zilber's Conjecture

Every subset of $\mathbb{C}$ definable in the structure $\mathbb{C}_{exp} := \langle \mathbb{C}; +, \cdot, -, exp, 0, 1 \rangle$ is either countable or co-countable.

[Also Koiran's conjecture: same conclusion for $\mathbb{C}_{1\text{-entire}} := \langle \mathbb{C}; +, \cdot, \{f\}_{f:\mathbb{C}\to\mathbb{C},\ \text{entire}} \rangle$.]

- Let K be either $\bar{\mathbb{C}}$ or $\bar{\mathbb{R}}$; $\mathcal{L} := \mathcal{L}(K_{exp})$.

- Let us consider existential formulas of $\mathcal{L}$:-

$$\Phi(\tilde{x}) : \quad \exists \bar{y} \bigwedge_i \bigvee_j \tau_{i,j}(\tilde{x}, \bar{y}) \,\Box_{i,j}\, 0$$

where the $\tau_{i,j}(\tilde{x}, \bar{y}) \equiv \tau_{i,j}(x_1, \ldots, x_m, y_1, \ldots, y_n)$ are terms of $\mathcal{L}$ and $\Box_{i,j}$ is $=$ or $\neq$.

Modulo field axioms + some simple axioms for exp, we have following equivalences:

$$\tau \neq 0 \iff \exists u \ \tau^2 - e^u = 0$$
$$(\tau_1 = 0 \vee \tau_2 = 0) \iff \tau_1 \cdot \tau_2 = 0$$
$$\tau_1(\cdots, \tau_2, \cdots) = 0 \iff \exists u \ (\tau_1(\cdots, u, \cdots) = 0 \wedge \tau_2 - u = 0).$$

$\therefore$ we may put our formula in the form:-

$$\Phi(\tilde{x}): \quad \exists \bar{y} \bigwedge_{j=1}^{p} F_j(\tilde{x}, \bar{y}) = 0 \qquad \text{------} (*)$$

where $F_j(\tilde{x}, \bar{y}) \in \Omega_{m,n} := \underbrace{\mathbb{Z}(\tilde{x}, e^{\tilde{x}}, \bar{y}, e^{\bar{y}})}$

polynomials over $\mathbb{Z}$ in $x_1, \ldots, x_m$, $e^{x_1}, \ldots, e^{x_m}$, $y_1, \ldots, y_n$, $e^{y_1}, \ldots, e^{y_n}$. (With possibly bigger n than before.)

Now write $\Phi(\tilde{x})$ as a finite union of $\exists_1$-formulas in <u>Khovanskii Normal Form</u> :—

$$K(\tilde{x}): \quad \exists \bar{y} \left(\bigwedge_{j=1}^{n} F_j(\tilde{x}, \bar{y}) = 0 \wedge \det\left[\frac{\partial(F_1, \ldots, F_n)}{\partial(y_1, \ldots, y_n)}\right](\tilde{x}, \bar{y}) \neq 0 \right)$$

(NB: each ring $\Omega_{m,n}$ is closed under partial differentiation.)

- This is non-trivial in the real case; it guarantees that there are only finitely many $\bar{y}$'s for each $\tilde{x}$ (with an upper bound independent of $\tilde{x}$). (Khovanski.)
- However, in the complex case it <u>is</u> trivial:

If $p < n$ in $(*)$, then the projection of
$\{ \bar{y} : \bigwedge_{j=1}^{p} F_j(\tilde{x}, \bar{y}) = 0 \}$ onto say, the y_1-complex line, will contain a non-empty open set, and hence some $q + is \in \mathbb{Q}[i]$.

So add the equations

$$e^{y_{n+1}} - 1 = e^{y_{n+2}} - 1 = e^{y_{n+3}} - 1 = 0$$

$$y_{n+3} \cdot y_1 - y_{n+2} - i \cdot y_{n+1} = 0$$

$$y_{n+3} - e^{y_{n+4}} = 0$$

(5 equations, but only 4 more variables).

[More interestingly: let $V \subseteq \mathbb{C}^n$ be a non-singular exponential-algebraic variety (defined using parameters). Then $\exists N > 0$ (independent of the parameters) and $\bar{y} \in V$ such that at least one coordinate y_i lies in $\frac{1}{N}\mathbb{Z}[i]$.]

Back to the real case: witnessing KNF-formulas

Suppose $\tilde{x} \in \mathbb{R}^m$, $y_1, \dots, y_n \in {}^*\mathbb{R}$ $(\subseteq {}^*\mathbb{C})$.

Let $k_0 = \mathbb{Q}(\tilde{x})$ $(\subseteq \mathbb{R} \subseteq {}^*\mathbb{R})$ and that $y_1, \dots, y_n$ witness a KNF-formula $K(\tilde{x})$.

We want to show that $y_1, \dots, y_n \in \mathrm{Fin}$.

In other words:

$$\underbrace{\mathrm{tr.deg}_{k_0}\left(k_0(y_1,\ldots,y_n,e^{y_1},\ldots,e^{y_n})\right)}_{K} \leq n.$$

$\therefore$ by VI, $\quad \mathrm{rank}\, v[K] \leq n$.

Easy combinatorial argument shows $v(e^{y_1}),\ldots,v(e^{y_n})$ must be $\mathbb{Z}$-dependent:

$$\sum_{j=1}^{n} a_j v(e^{y_j}) = 0 \qquad (a_1,\ldots,a_j \in \mathbb{Z},\text{ not all zero})$$

– i.e. $\quad \log\left|e^{\sum_{j=1}^n a_j y_j}\right| \in \mathrm{Fin} \setminus \mu$.

– i.e. (!!) $\quad \sum_{j=1}^{n} a_j y_j \in \mathrm{Fin}$.

So may as well suppose that $y_1 \in \mathrm{Fin}$.

We want to repeat the argument on $y_2,\ldots,y_n$.

If st.part(y_1) is transcendental over $\mathbb{Q}(\pi)$ $(= k_0)$, then $\mathrm{tr.deg}_{k_0}(\text{res. field } K) \geq 1$, so $\mathrm{rank}\, v[K] \leq n-1$ (by VI), and we <u>may</u> repeat the argument.

Otherwise we are <u>stuck</u> unless we can reduce $\mathrm{trdeg}_{k_0} K$, which would contradict Schanuel's conjecture!!

Solution: expand the language of fields so that y_1 and e^{y_1} are dependent and VI still holds.

obviously cannot expand to full exponentiation, but:—

Theorem 1 (AJW / van den Dries)

$\langle \bar{\mathbb{R}} ; \exp\restriction[0,1] \rangle$ (and $\langle \bar{\mathbb{R}} ; \exp\restriction[0,1] ; \sin\restriction[0,1] \rangle$) have model complete theories and are polynomially bounded, o-minimal structures with field of exponents $\mathbb{Q}$).

Theorem 2 (AJW / v.d.D / v.d.D - Lowenberg / Miller / Tyne)

Let $\tilde{\mathbb{R}}$ be a poly. bnd., o-min expansion of $\bar{\mathbb{R}}$ with field of exponents $\mathbb{Q}$. Let $k_0 \leq K \leq {}^*\mathbb{R}$ (considered as an $\mathcal{L}(\tilde{\mathbb{R}})$-structure) with $k_0 \subseteq \mathcal{F}in$. Then there exists a copy k of the residue field of K with $k_0 \leq k \leq K$ (and $k \subseteq \mathcal{F}in$) and, further:

(VI) $\quad \dim(K/k_0) \geq \operatorname{rank}(v[K]) + \dim(k/k_0)$

($\dim(X)$ = dimension associated with 0-definable closure in ${}^*\mathbb{R}$.)

<u>Corollary</u> : — $y_1, y_2, \ldots, y_n \in Fin$. More generally, If $K(\tilde{x}) := \exists \bar{y}\, \phi(\tilde{x}, \bar{y})$ is a KNF formula, $\tilde{x} \subset {}^*\mathbb{R}$, and if $\bar{y} \subset {}^*\mathbb{R}$ witnesses $K(\tilde{x})$, then such a witness can be found in any real closed field $M \subseteq {}^*\mathbb{R}$ with $\mathbb{Q}(\tilde{x}) \subseteq M$ and $M \equiv \mathbb{R}_{exp}$. Hence $Th(\mathbb{R}_{exp})$ is model complete.

<u>*TF</u>

<u>The complex case</u>

<u>Conjecture A</u>

Let $K^c(\tilde{z})$ be in KNF:

$$\exists \bar{w} \left(\bigwedge_{j=1}^{n} F_j(\tilde{z}, \bar{w}) = 0 \wedge \det(\cdots)(\tilde{z}, \bar{w}) \neq 0 \right)$$

$\left(F_j(\tilde{z}, \bar{w}) \in \mathbb{Q}(i)[\tilde{z}, e^{\tilde{z}}, \bar{w}, e^{\bar{w}}] \right)$

Let $\tilde{a}, \tilde{a}' \in \mathbb{C}^m$ and assume that the line $[\tilde{a}, \tilde{a}']$ contains only generic points. Then any solution $w_1(\tilde{z}), \ldots, w_n(\tilde{z})$ around $\tilde{a}$ can be analytically continued to $\tilde{a}'$.

This conjecture implies Zilber's conjecture. (In fact, even for $L_{\infty\omega}$-formulas.)

"generic" here refers to the pregeometry $\mathcal{K}(\cdot)$ determined by KNF-formulas: if $\bar{w} = w_1, \ldots, w_n$ satisfies

$$\bigwedge_{j=1}^{n} F_j(\tilde{z}, \bar{w}) = 0 \wedge \det(\cdots)(\tilde{z}, \bar{w}) \neq 0$$

then, by definition, $w_1 \in \mathcal{K}(\tilde{z})$.

- The conjecture A implies that any two generic elements $z_1, z_2 \in \mathbb{C}$ satisfy the same $\mathcal{L}_{\omega\omega}$ formulas (for $\mathcal{L} = \mathcal{L}(\mathbb{C}_{exp})$). So a definable $X \subseteq \mathbb{C}$ either contains no generics (so X is countable) or every generic (X is co-countable).

- To prove the conjecture, suppose we can continue $\langle w_1(\tilde{z}), \ldots, w_n(\tilde{z}) \rangle$ for $\tilde{z} \in [\tilde{a}, \tilde{a}')$ but $\bar{w}(\tilde{z})$ is unbounded as $\tilde{z} \to \tilde{a}'$. Go to ${}^*\mathbb{C}$. Choose $\tilde{z} \in {}^*[\tilde{a}, \tilde{a}')$ with $\tilde{a}' - \tilde{z} \in \mu({}^*\mathbb{C})^m$, and $w_1(\tilde{z}), \ldots, w_n(\tilde{z}) \in {}^*\mathbb{C}$ with, say, $w_1(\tilde{z}) \notin \mathrm{Fin}({}^*\mathbb{C})$.

Since st. part $(\tilde{z})$ $(= \tilde{a}')$ is generic,

$\mathbb{Q}(i)(\tilde{z}) \subseteq \mathrm{Fin}({}^*\mathbb{C})$ (Rouché's theorem).

$\mathbb{Q}(i)(\tilde{z}) = k_0$

Now apply valuation theory to $k_0(\bar{w}(\tilde{z}))$.

<u>Theorem</u>

The pregeometry $\mathcal{K}(\cdot)$ can be extended to ${}^*\mathbb{C}$ so that the following holds:

Let k_0 be any $\mathcal{K}(\cdot)$-closed subset of ${}^*\mathbb{C}$ with $k_0 \subseteq \mathrm{Fin}$. Let $a_1, \ldots, a_N \in {}^*\mathbb{C}$ and let $M = \mathcal{K}(k_0 \cup \{a_1, \ldots, a_N\})$. Then there exists a $\mathcal{K}(\cdot)$-closed set $k \subseteq \mathrm{Fin}$ with $k_0 \subseteq k$ which is a copy of the residue field of M. Further

$N \geq \mathcal{K}(\cdot)\text{-dim}(M/k_0) \geq \mathrm{rank}\, v[M] + \mathcal{K}(\cdot)\text{-dim}(k/k_0)$.

Doesn't get what we want – i.e. $\bar{w}(\tilde{z}) \in \mathrm{Fin}({}^*\mathbb{C})^n$

– because $e^{w_i} \in \mathrm{Fin} \setminus \mu \not\Rightarrow w_i \in \mathrm{Fin}$.

However,

$$e^{w_1} \in \mathcal{F}in \setminus \mu \quad \text{and} \quad e^{iw_1} \in \mathcal{F}in \setminus \mu$$

does imply $w_1 \in \mathcal{F}in$, and so we do get:

Theorem

$\langle \mathbb{C}, +, \cdot, -, 0, 1, z \mapsto z^i \rangle$ is quasi-minimal.

(But this structure was already known to be quite tame – even with a predicate for $\mathbb{R}$ – by results of C. Miller)

THANKS EVERYONE !!