Properties of Bootstrap Moran's I for Diagnostic Testing a Spatial Autoregressive Linear Regression Model

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Abstract

Moran's I or Cliff-Ord test statistic has been widely used for diagnostic testing of spatial correlation in a linear regression model with or without spatial autoregressive lags. The latter simple model can be easily estimated with OLS, while the former spatial lag model relies on maximum likelihood or the instrumental variables method. Specification testing for spatial autocorrelation is typically performed with the asymptotic distribution of Moran's I test statistic, which depends on the normality assumption of the model. For many real world applications, the asymptotic theory of the Moran test may not be applicable because the classical normality assumption is rarely satisfied. In this paper, we apply residual-based wild bootstrap methods for hypothesis testing of spatial autocorrelation in a linear regression model. For specification tests of a spatial autoregressive linear regression model, our simulation and bootstrap computations are presented with the consistent instrumental variables or 2SLS estimation method. Based on Moran's I test statistic, the empirical size and power of bootstrap and asymptotic tests for spatial correlation are evaluated and compared. Under the normality assumption of the model, the performance of the bootstrap test is equivalent to or better than that of the asymptotic test. For more realistic heterogeneous non-normal models, the applicability of asymptotic tests is questionable. Bootstrap tests have shown superiority in smaller size distortion and higher power when compared to asymptotic counterparts, especially for cases with a small sample and dense spatial contiguity. Our Monte Carlo experiments indicate that based on Moran's I test statistic, the bootstrap method is an effective alternative to the theoretical asymptotic approach when the classical normality assumption is not warranted.

Keyword: Moran's I, Monte Carlo, size distortion, power, wild bootstrap
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1. Introduction

Spatial econometrics is a specialized subfield of econometrics that studies spatial characteristics and spatial interaction of cross section and panel data. The characteristics of spatial heterogeneity and spatial autocorrelation are the main features of the spatial data. Theoretical and applied spatial econometrics put forth the analysis of spatial interaction across heterogeneous and dependent data observations. A popular model of spatial interactions was developed by Cliff and Ord (1973, 1981), in which spatial lags are incorporated in the regressand, regressors, and disturbance terms of the linear regression equation. Hypothesis testing and estimation of the Cliff-Ord type models have been developed and applied in economics, regional science and geography (see Anselin (1988), Anselin and Rey (1991), Anselin and Florax (1995), Anselin, Bera, Florax, and Yoon (1996), among others). More recent contributions of spatial econometric analysis include Anselin and Kelejian (1997), Kelejian and Prucha (1998, 1999, 2001, 2009), Lee (2003, 2004, 2007), and many others.

In spatial data analysis, hypothesis testing for spatial correlation is a common practice for the identification and estimation of a spatial econometric model. Moran's I or Cliff-Ord test statistic has been widely used for diagnostic testing of spatial correlation in a linear regression model with or without autoregressive spatial lags. The Moran test for the latter simple model is based on OLS residuals of a linear regression model and can be easily computed[1]. Due to the presence of a spatial lagged regressand, the OLS estimator is no longer valid for a spatial autoregressive model, relying instead on maximum likelihood (ML) or the instrumental variables 2SLS method. Therefore, for hypothesis

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[1] Although Moran's I test statistic is a simple and popular test for spatial dependence, it does not distinguish the spatial correlation in the variable or in the error. LM test statistics for spatial lag and for spatial error supplement the Moran's I statistic for specification test and model validation (see Anselin, Bera, Florax, and Yong (1996) for more details).

testing of spatial dependence in a spatial autoregressive model, the relevant test statistics must be based on the regression residuals of the consistent ML or 2SLS estimator (see Anselin and Kelejian (1997), Kelejian and Prucha (2001)). This paper focuses on diagnostic testing of spatial correlation in a spatial autoregressive lag model (or "SAR model" hereafter) using Moran's I statistic based on 2SLS regression residuals.

The available literature concerning properties of diagnostic testing for spatial correlation in the SAR model framework is sparse. Kelejian and Prucha (2001) derived a large sample distribution of the Moran's I test statistic for a variety of important models including the SAR model. Under the assumption of i.i.d. innovations, Moran's I statistic follows the normal distribution asymptotically. In other words, when the i.i.d. assumption is violated, it is questionable whether testing for spatial correlation can be based on the theoretical normal distribution of Moran's I statistic. Anselin and Kelejian (1997) indicate that Moran's I test based on its asymptotic theory works poorly in the SAR model framework. For many real world applications, the asymptotic theory of Moran's I test may not be applicable because the classical i.i.d. assumption is rarely satisfied. In our previous work (Lin, Long, and Ou (2008)) examining the properties of the Moran test using bootstrap methods in a linear regression model with non-normal innovations, we found that the bootstrap test is an effective alternative to the asymptotic test in terms of size and power performance. In this paper, we study and evaluate the empirical size and power of the bootstrap test based on Moran's I statistic in the SAR model framework. Evidences from extensive Monte Carlo experiments show that our results compare favorably with bootstrap methods for diagnostic testing of spatial correlation in the SAR model framework.

This paper consists of four sections. In the first section, we estimate the SAR model using the 2SLS method and derive Moran's I statistic to test for spatial correlation in the SAR model. After a brief review of concepts of the spatial bootstrap test, the second section proposes the analytical framework of studying properties of spatial bootstrap tests based on the P-value of Moran's I statistic. The third section describes the simulation design and reports Monte Carlo experiments of our current study. The final section is the conclusion.

2. Spatial Autoregressive Model and Moran's I Test Statistic

In this section, we give a brief description of the SAR model and outline the Moran test procedure for spatial correlation. First, we present the 2SLS estimation for the SAR model. Then, we summarize the asymptotic distribution of Moran's I statistic in the SAR model framework.

2.1 SAR Model

We consider the following cross sectional SAR model:

$$\begin{aligned} y &= \lambda W y + X\beta + \varepsilon \\ &= Z\delta + \varepsilon \qquad (1) \\ \varepsilon_i &\sim iid(0,\sigma^2), \ |\lambda| < 1 \end{aligned}$$

where $Z = (Wy, X)$ and $\delta = (\lambda, \beta')'$. Here y is an $N \times 1$ vector of data observations on the dependent variable; λ is the spatial lag parameter, which is assumed to be less than one in absolute value; W is an $N \times N$ non-stochastic weights matrix with zero diagonal and row-standardized, which are known a priori; X is a fixed $N \times k$ matrix of data observations on the explanatory variables; β is a $k \times 1$ vector of regression coefficients; ε is an $N \times 1$ vector of regression disturbances.

The specifications in (1) can be expressed in the reduced form as:

$$y = (I_n - \lambda W)^{-1}(X\beta + \varepsilon)$$

Furthermore,

$$\begin{aligned} E[(Wy)\varepsilon'] &= W(I_n - \lambda W)^{-1} E[(X\beta + \varepsilon)\varepsilon'] \\ &= \sigma^2 W(I_n - \lambda W)^{-1} \\ &\neq 0 \end{aligned}$$

Obviously, the spatial lagged dependent variable Wy is random and correlated with the disturbance term ε. The model cannot be estimated by the OLS method. If the disturbances are known to follow a normal distribution, the ML method is efficient. Quasi ML (or QML) method can be used even if the model distribution is not normal (see Lee (2004)). However, computational complexities involved with the inverse and determinant evaluation of the Jacobian matrix create a practical difficulty in using the ML or QML method even if the sample size is only moderate (see Kelejian & Prucha, (1999)). These complexities can be especially overwhelming if the ML or QML method is used in a Monte Carlo simulation. Nevertheless, the SAR model can be consistently estimated by the 2SLS or GMM method with proper instruments. The 2SLS or GMM method is computationally simpler than the QML method and is free of distributional assumption (see Kelejian & Prucha (1998, 1999), Lee (2003, 2007)). It is the current trend to use 2SLS or GMM methods for a spatial model estimation, particularly a Monte Carlo simulation. In this study, we implement the 2SLS method to estimate the SAR model according to the following steps:

(i) Choose the instrument variables matrix H with full column rank. It is composed of a subset of the linearly independent columns of $(X, WX, W^2X, \cdots)$. We usually let H be the linearly independent columns of (X, WX, W^2X).

(ii) Regress Wy on H, and obtain fitted values $\hat{Wy} = PWy$. Here $P = H(H'H)^{-1}H'$.

(iii) Obtain the consistent estimator of $\delta = (\lambda, \beta')'$ by regressing y on $\hat{Z} = (\hat{Wy}, X)$. Namely, $\hat{\delta} = (\hat{Z}'\hat{Z})^{-1}\hat{Z}\, y$. It is equivalent to $\hat{\delta} = (\hat{Z}'Z)^{-1}\hat{Z}\, y$ or $\hat{\delta} = (Z'PZ)^{-1}Z'Py$.

The consistent estimator $\hat{\delta}$ is sensitive to the choice of the spatial contiguity matrix in addition to the classical i.i.d. assumption. For example, the 2SLS estimator vector $\hat{\delta}$ is inconsistent when the spatial weights matrix W has zero diagonal elements and other elements are equal to $\frac{1}{N-1}$ (Kelejian & Prucha (2002)). In others words, 2SLS estimation methods are not efficient with the equal spatial weights.

We now obtain the regression residuals based on the 2SLS consistent estimator $\hat{\delta} = (\hat{\lambda}, \hat{\beta}')'$, from which Moran's I statistic is computed.

2.2 Moran's I Statistic

The most popular testing procedure for spatial correlation is the one based on Moran's I statistic (Moran (1950)). Though it has a long history of applications in linear regression models, the Moran test has only recently been applied to the SAR model. Anselin (1988) presented a large sample distribution of this statistic based on ML residuals (see also Anselin, Bera, Florax, and Yoon (1996)). Kelejian and Prucha (2001) derived a general result based on 2SLS residuals (see also Anselin and Kelejian (1997), Pinkse (2004)). According to Kelejian and Prucha (2001), we summarize the asymptotic distribution of Moran's I statistic in the SAR model framework.

For a given spatial weight matrix W, Moran's I test statistic for the SAR model is given by

$$I = \frac{\hat{\varepsilon}' W \hat{\varepsilon}}{\hat{\varepsilon}' \hat{\varepsilon}} \qquad (2)$$

where $\hat{\varepsilon} = y - \hat{\lambda} Wy - X\hat{\beta}'$ is the vector of 2SLS residuals. Under some regularity assumptions (see Kelejian & Prucha (2001)), Moran's I statistic in (2) is shown to be asymptotically normal, specifically,

$$\frac{I}{\sqrt{Var(I)}} \sim N(0,1) \qquad (3)$$

where

$$Var(I) = \frac{1}{N^2}\{trace(WW + W'W) + \frac{\gamma}{\hat{\sigma}^2}\},\ \gamma = \hat{\varepsilon}'(W + W')Z(Z'PZ)^{-1}Z'(W + W')\hat{\varepsilon},\ \hat{\sigma}^2 = \frac{\hat{\varepsilon}'\hat{\varepsilon}}{N}.$$

Therefore, based on 2SLS residuals of the SAR model, we can theoretically apply the above Moran's I statistic to test for spatial correlation, provided the model error is normally independently distributed.

3. Size and Power of the Spatial Bootstrap Test

Instead of relying on the asymptotic theory of the Moran test, we suggest the use of the bootstrap method for diagnostic testing of spatial correlation in the SAR model. Based on the empirical distribution of the Moran's I test statistic, the performance of the bootstrap test is evaluated. In particular, following the similar design and procedure of Anselin and Kelejian (1997) and Lin, Long, and Ou (2008), the finite sample properties of the bootstrap Moran test are examined from a large number of Monte Carlo simulation experiments.

In the following sections, we review the bootstrap methods suitable for testing spatial dependence in a spatial econometric model. We outline the simulation and bootstrap procedure to construct the empirical distribution of the Moran's I test statistic. By examining the empirical and asymptotic distributions of the test statistics, the size and power of the bootstrap test can be evaluated and compared with those of the asymptotic test.

3.1 Spatial Bootstrap Test Based on Moran's I Statistic

In many real world applications, hypothesis testing based on the asymptotic theory may not be applicable because classical model assumptions such as homoscedasticity and normality are rarely satisfied. In 1979, Efron introduced a resampling procedure, named the bootstrap, for estimating the distribution of model parameters and test statistics. The bootstrap method is a useful nonparametric alternative to the theoretical asymptotic approach for model estimation and hypothesis testing. It has been successfully applied in time series and panel data econometrics (see Davidson and MacKinnon (2006), Chang (2004), Park (2003)).

For a spatial econometric model, the asymptotic theory of Moran's I statistic in the SAR model is not valid when the model error is heteroscedastic or non-normally distributed. A bootstrap technique used to overcome this difficulty is called the spatial bootstrap test. The spatial bootstrap test is defined to be a residual-based wild bootstrap method for a spatial econometric model, in which the spatial structure of the cross section data is preserved during bootstrap simulation. Given fixed regressors and an exogenously defined spatial weights matrix, the spatial data structure is maintained by bootstrapping the regression residuals. Lin, Long and Ou (2008) found that a spatial bootstrap test based on the OLS residuals is an effective alternative to the theoretical asymptotic approach when the classical normality assumption is violated. In this study, the spatial bootstrap test based on Moran's I statistic is used to test for spatial correlation in the SAR model.

For model evaluation, small sample properties of the parameter estimators and test statistics are evaluated using Monte Carlo simulations. In the following, we describe the spatial bootstrap test based on a typical Monte Carlo simulation experiment as follows:

(i) For each i-th Monte Carlo experiment ($i = 1, \cdots, M$), given the fixed regressors X and spatial weights matrix W, we consider the data generating process (DGP) μ_i or a pair of the data observation (y, X). The data is used to estimate the SAR model (1) using the 2SLS method. Then, with the 2SLS estimator $\hat{\delta} = (\hat{\lambda}, \hat{\beta}')'$ and the residual vector $\hat{\varepsilon} = y - Z\hat{\delta} = y - \hat{\lambda} W - X\hat{\beta}$, we compute the Moran's I test statistic $\hat{I}$ according to (2).

(ii) Re-scale and re-center the 2SLS residual $\hat{\varepsilon}_k$:

$$\tilde{\varepsilon}_k = \sqrt{\frac{N}{N-1}} \left[\frac{\hat{\varepsilon}_k}{\sqrt{1-h_k^2}} - \frac{1}{N} \sum_{l=1}^{N} \frac{\hat{\varepsilon}_l}{\sqrt{1-h_l^2}} \right], k = 1,2,\cdots,N \qquad (4)$$

Where h_k (or h_l) is the *k*-th (or *l*-th) diagonal element of the hat matrix $Z(\hat{Z}'Z)^{-1}\hat{Z}'$ based on the 2SLS estimation method.

(iii) We obtain a bootstrap sample $\tilde{e}$ by N random draws with replacement from the elements of $\tilde{\varepsilon}$. It is evident that the *wild* bootstrap is more effective in a model with general heteroscedasticity (see Wu (1986), Liu (1988), Davidson and Flachaire (2008)). Thus, the spatial bootstrap test is based on the wild bootstrap of 2SLS residuals in the SAR model. Let $\hat{e}_k = \tilde{e}_k \cdot v_k$, ($k = 1,2,\cdots,N$). The popular choices for the distribution of the random variable v_k are the following two-point distributions (see Mammen (1993), Davidson and Flachaire (2008)):

First, $$v_k = \begin{cases} -1 & \text{with probability } 0.5 \\ 1 & \text{with probability } 0.5 \end{cases} \qquad (5)$$

Second, $$v_k = \begin{cases} -(\sqrt{5}-1)/2 & \text{with probability } \mathrm{p} = (\sqrt{5}+1)/(2\sqrt{5}) \\ (\sqrt{5}+1)/2 & \text{with probability } 1-\mathrm{p} \end{cases} \qquad (6)$$

We call the wild bootstrap method corresponding to (5) the *symmetric* wild bootstrap, and that corresponding to (6) the *asymmetric* wild bootstrap.

(iv) During each i-th Monte Carlo experiment, we obtain the *j*-th bootstrap DGP $\mu^*_{i,j}$ from μ_i and the corresponding bootstrap sample is $(\tilde{y}, X)$ where $\tilde{y} = \hat{\lambda} Wy + X\hat{\beta} + \hat{e}$. We note that X is fixed and W is exogenous. With the 2SLS estimator $\delta^* = (\lambda^*, \beta^{*\prime})'$ of the SAR model, the residual vector is $e^* = y - \lambda^* Wy - X\beta^*$. Based on (2), we compute the Moran's I test statistic, and denote $\hat{I}^*_{i,j}$.

(v) Repeat (iii) and (iv) for B times. For the i-th Monte Carlo experiment with the original data (y, X), the Moran's I statistic is $\hat{I}_i$; with its B bootstrap samples $(\tilde{y}, X)$, the corresponding bootstrap test statistics are $\hat{I}^*_{i,1}, \hat{I}^*_{i,2}, \cdots, \hat{I}^*_{i,B}$.

After a large number of Monte Carlo simulation experiments are performed based on the above bootstrap procedure, there are M sets of bootstrap Moran's I test statistics ($\hat{I}_i, \hat{I}^*_{i,1}, \hat{I}^*_{i,2}, \cdots, \hat{I}^*_{i,B}$, i = 1,2,…M). From the empirical distribution of these Moran's I test statistics, we evaluate and compare the performance of the asymptotic and bootstrap tests.

3.2 P-Value of the Spatial Bootstrap Test Statistic

P-value is the probability of the null hypothesis being rejected. Generally, we compute P-value for hypothesis testing based on the asymptotic distribution of the test statistic. Since the bootstrap test is a nonparametric process in nature, its P-value is computed from the empirical distribution of the test. For hypothesis testing, P-value can provide more information than the test statistic itself (see Davidson and MacKinnon (1999, 2006)).

The Moran test for spatial correlation is a two-tail test. However, it may not be symmetric. For the i-th Monte Carlo experiment with DGP μ_i, i = 1,2,…M, the equal-tail P-value of the bootstrap Moran's I test statistic can be expressed as:

$$P^*(\hat{I}_i) = 2\min\left(\frac{1}{B}\sum_{j=1}^{B} I(\hat{I}^*_{i.j} \le \hat{I}_i), \frac{1}{B}\sum_{j=1}^{B} I(\hat{I}^*_{i,j} > \hat{I}_i)\right) \qquad (7)$$

where $I(\cdot)$ denotes the indicator function, which is equal to 1 if its argument is true and 0 otherwise. Given a nominal level of significance α, we compare $P^*(\hat{I}_i)$ with α. For example, $\alpha = 5\%$. Specifically, when $P^*(\hat{I}_i) < \alpha$, we reject the null hypothesis of no spatial dependence. Otherwise, it can not be rejected. For notational compatibility, we denote $P(\hat{I}_i)$ the P-value of the Moran's I test statistic $\hat{I}_i$ based on the asymptotic normal distribution.

3.3 Size Distortion of the Spatial Bootstrap Test

Based on the bootstrap Moran's I test statistic, we expect that the count of $P^*(\hat{I}_i) < \alpha$ or the rejection probability is small in order to accept a true hypothesis of no spatial correlation. When a bootstrap test wrongfully rejects the true null, the rejection probability is $\frac{1}{M}\sum_{i=1}^{M} I(P^*(\hat{I}_i) < \alpha)$. The size distortion of the test is the difference between the size and the nominal level of significance α. The smaller the size distortion is, the better the spatial bootstrap test performs. We have

$$\text{Size Distortion } = \frac{1}{M}\sum_{i=1}^{M} I(P^*(\hat{I}_i) < \alpha) - \alpha \qquad (8)$$

where $I(\cdot)$ denotes the indicator function of rejection probability. That is, when $P^*(\hat{I}_i) < \alpha$, we reject the null hypothesis of no spatial correlation and $I(P^*(\hat{I}_i) < \alpha) = 1$. Otherwise, we cannot reject the null and $I(P^*(\hat{I}_i) < \alpha) = 0$.

3.4 Power of the Spatial Bootstrap Test

Similarly, if the 2SLS residuals of the SAR model were contaminated with spatial correlation, we expect that hypothesis testing can identify the spatial correlation with a large probability of rejecting the false null hypothesis. By definition, this rejection probability is the *power* of the test.

Suppose that the null hypothesis is false and if $P^*(\hat{I}_i) \geq \alpha$, we fail to reject it. This is called a Type II error with the probability $\beta = \frac{1}{M}\sum_{i=1}^{M} I(P^*(\hat{I}_i) \geq \alpha)$. Therefore, the power of the test is one minus the probability of Type II error. That is,

$$Power = 1 - \beta = \frac{1}{M}\sum_{i=1}^{M} I(P^*(\hat{I}_i) < \alpha) \qquad (9)$$

Obviously, the power of a test depends on the alternative hypothesis under consideration. For simplicity, in this study we consider spatial autocorrelation in the model disturbances – the spatial autoregressive model with autoregressive disturbance (or SARAR) model. Other alternatives are possible, such as moving average disturbances or mixed and higher order models.

With a large number of Monte Carlo simulations, we can compute the size distortion and power of the spatial bootstrap test based on (8) and (9) respectively. Then, the performance of the spatial bootstrap test can be evaluated. In general, a smaller size distortion and larger power are preferred. It is clear that the comparison and evaluation of the spatial bootstrap and asymptotic tests rely on the simulation design and strategy of the Monte Carlo experiments described below.

4. Monte Carlo Experiments of the Spatial Bootstrap Test

In this section we first present the design of Monte Carlo experiments for the SAR model framework. Then we describe the simulation procedure to evaluate the size and power of the spatial bootstrap test for spatial correlation in the SAR model. Finally, we report the simulation results.

4.1 Experimental Design

To test for spatial dependence in the SAR model, the null hypothesis assumes no spatial correlation in the 2SLS residuals. The null model is the maintained SAR model without spatial autocorrelation in the model disturbances. Although there are many possible alternatives of interest, for simplicity we assume the SARAR model defined by $y = \lambda Wy + X\beta + (I_N - \rho W)^{-1}\varepsilon$. Depending on the size or power evaluation, the data generating process for the Monte Carlo experiments assume either the null SAR model or the alternative SARAR model[2].

Based on the experience of previous simulation studies in related models (see Anselin, Bera, Florax, and Yoon (1996), Anselin and Kelejian (1997), Davidson and MacKinnon (2002)), we perform 5000 Monte Carlo experiments for each model design with the following setup of variables and parameters in the SAR model framework:

(i) The spatial weights matrices under consideration include one real irregular-spaced first-order contiguity matrix Wmat of Anselin (1998)[3] and two hypothetical regular-spaced contiguity matrices of Rook (4 neighbors) and Queen (8 neighbors) configuration, respectively. These spatial contiguity matrices are properly row-standardized with zero diagonal. Different sample sizes are considered as well: N = 36, 49, 81, 121.

(ii) X is the data matrix of three independent variables, where the first two are drawn from a uniform (0,10) distribution and the third is a constant term 1. We superimpose an outlier in the first independent variable[4]. The corresponding regression parameter vector is $\beta = (1,1,1)'$.

(iii) Under the null hypothesis of a SAR model, we consider four cases of spatial lag parameters: $\lambda = -0.7, -0.3, 0.3, \text{and } 0.7$. When the alternative SARAR model is considered, we assume that the spatial autocorrelation parameter ρ ranges from -0.9 to 0.9 with an interval of 0.1.

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[2] An interesting alternative is the spatial autoregressive with moving average disturbances (or the "SARMA" model) defined by $y = \lambda Wy + X\beta + (I_N + \theta W)\varepsilon$. Our extensive Monte Carlo simulations include this alternative, but do not find any major differences from the SARAR model.

[3] Spatial weights matrix Wmat was first used in Anselin (1988) for a crime rate study of 49 neighborhoods in Columbus, Ohio.

[4] As in Davidson and MacKinnon (2002), the second observation of the first independent variable is inflated 10 times the original value. The abnormality of the first independent variable is used to magnify the effect of heteroscedasticity in the model.

(iv) For the model disturbance term ε , we consider two distributions: ε_1 and ε_2 . ε_1 is named the *heteroscedastic disturbance* and is defined by the product of a standard normal variate and the first independent variable with outlier (see Davidson and MacKinnon (2002)). ε_2 is a weighted average of $\chi^2(1)$ random variate and $t(2)$ random variate with a uniformly distributed random weight between 0 and 1. This is a stochastic mixture of two non-normal distributions, which generates a random variable with unknown distribution. We call this a *mixture disturbance.*

(v) Based on the above variables and parameters setup, the dependent variable y is generated as $y = \lambda Wy + X\beta + \varepsilon$ for the null SAR model. For the alternative SARAR model $y = \lambda Wy + X\beta + (I_N - \rho W)^{-1}\varepsilon$ when $\rho \neq 0$.

When we study the size distortion of the spatial bootstrap test, the DGP is based on the null SAR model. For power studies, the DGP depends on the alternative SARAR model.

4.2 Simulation Procedure

To study the properties of the spatial bootstrap test in relation to its asymptotic counterpart, we describe the step-by-step procedure of the Monte Carlo simulation as follows:

(i) According to the experimental design, the regressors *X*, spatial contiguity weights matrix *W*, regression parameter vector β , and spatial correlation parameters (λ , ρ) are given, where *X* and *W* are assumed to be fixed in all Monte Carlo experiments.

(ii) A model disturbance term (ε_1 or ε_2) is generated randomly, according to the model assumption under consideration. That is, ε_1 is a heteroscedastic disturbance and ε_2 is a non-normal mixture disturbance.

(iii) When we study the size distortion of the spatial bootstrap test, the dependent variable y is generated as $y = \lambda Wy + X\beta + \varepsilon$ for the null SAR model. When the power of the spatial bootstrap test is considered, the dependent variable is generated as $y = \lambda Wy + X\beta + (I_N - \rho W)^{-1}\varepsilon$ for the alternative SARAR model.

(iv) The spatial bootstrap test is computed according to Section 3.1. We obtain 1+*B* Moran's I statistics for testing spatial correlation using a simulated sample (y, X) and the corresponding B bootstrap samples $(\tilde{y}, X)$. These are, respectively, $\hat{I}_i$ and $\hat{I}^*_{i,1}, \hat{I}^*_{i,2}, \cdots, \hat{I}^*_{i,B}$.

(v) Based on the empirical distribution of the Moran's I statistics, we compute the equal-tail P-value of the spatial bootstrap test's Moran's I statistics as described in Section 3.2. According to (7), $P^*(\hat{I}_i)$ and $P(\hat{I}_i)$ are the respective P-value of the spatial bootstrap test and the asymptotic test.

(vi) Repeat steps (ii)-(v) for all M experiments, and with the corresponding DGPs we obtain M sets of P-values for the spatial bootstrap test and asymptotic test, $P^*(\hat{I}_i)$ and $P(\hat{I}_i)$, $i = 1,2,\cdots,M$ with $M = 5000$.

(vii) When the DGP is based on the null SAR model, the size distortions of the spatial bootstrap and asymptotic tests are computed according to (8). When the DGP is based on the alternative SARAR model, the powers of the spatial bootstrap and asymptotic tests are computed according to (9).

From a large number Monte Carlo experiments, the size and power of the spatial bootstrap test is evaluated and compared with those of the asymptotic test.

4.3 Simulation Results

It is expected that, under classical normality assumptions of the model, the performance of the bootstrap test is equivalent to or better than that of the asymptotic test in terms of size and power (see also Lin, Long, and Ou (2008)). We are interested in two more realistic model structures: the heteroscedastic disturbance ε_1 and the mixture disturbance ε_2 with unknown distribution. We apply two residual-based wild bootstrap methods: symmetric wild bootstrap and asymmetric wild bootstrap as described in Section 3.1, in which the asymmetric wild bootstrap method is shown to be a more effective method for non-normal models. Therefore, in the following, we present only the findings for these models using the asymmetric wild bootstrap method[5]. In particular, we focus on two measures of the model performance: (i) a comparison of the size distortion of the spatial bootstrap test and asymptotic test under the true null SAR model; (ii) a comparison of the relative power of the tests against the alternative SARAR model[6].

4.3.1 Size Distortion of the Spatial Bootstrap Test

Under the null hypothesis of no spatial autocorrelation in the SAR model, we investigate size distortion of the spatial bootstrap test for two non-normal model structures and three spatial weights matrices. The two model structures are the heteroscedastic disturbances ε_1 and the mixture disturbances ε_2. Three spatial weights configurations under consideration are Anselin's Wmat matrix, Rook matrix and Queen matrix. To be compatible with the sample size of Anselin's crime study (1988), our results of size distortion for spatial bootstrap and asymptotic tests are based on the sample size *N*=49.

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5 Because of space limitation, the lengthy output of the simulation results for the classical normal models and for the non-normal models with symmetric wild bootstrap method are not reported here. They are available upon request.

6 In this paper, all simulation and bootstrap computations are run using GPE2/GAUSS7.0 (see Lin (2001)) on a Pentium 4 2.4Ghz Windows-XP system.

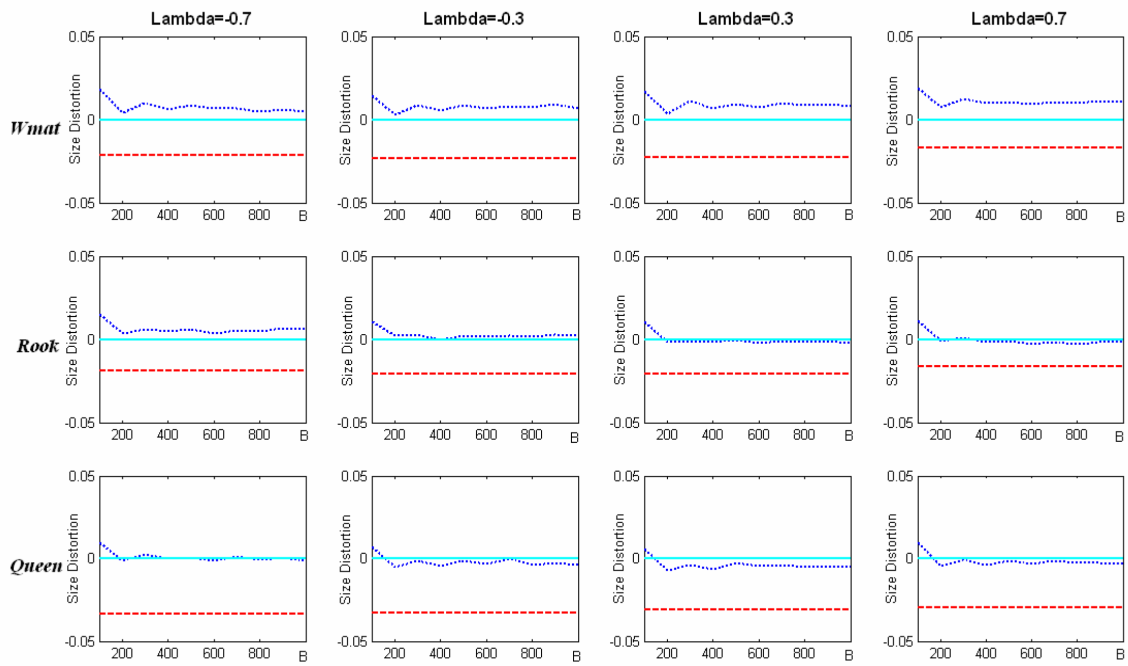

Figure 1. Size Distortion of the Spatial Bootstrap and Asymptotic Tests: SAR Model with Heteroscedastic Disturbances, N=49, M=5000

The graphical presentations of the size distortion for two model structures using the asymmetric wild bootstrap method are given in Figure 1 and Figure 2, respectively. We combine the plots for three spatial contiguity matrices over four values of spatial lag parameters (λ or Lambda = -0.7, -0.3, 0.3, 0.7). These plots are arranged into a grid table of three rows and four columns.

Figure 1 portrays the size distortion of both spatial bootstrap and asymptotic tests for models with heteroscedastic disturbances ε_1. In relation to the number of bootstrap replications (B) on the horizontal axis, size distortion is shown on the vertical axis. The dotted curve and the dashed line represent the size distortion of the spatial bootstrap and asymptotic tests, respectively. The solid horizontal line is the ideal size distortion of zero. The closer the dotted curve (or the dashed line) is to the zero line, the better the performance of the bootstrap test (or the asymptotic test) is. The detailed numerical results are given in the Appendix Tables A.1-A.3.

We observe that the dashed lines for asymptotic tests are far below zero for all three different spatial weights matrices and four selected spatial lag parameters under consideration. The size distortions of the asymptotic test are rather large relative to the 5% nominal level of significance. It indicates a negative bias of the asymptotic test in all cases. The asymptotic test under-rejects the null hypothesis of no spatial correlation for all irregular and regular lattice spatial weights matrices. It implies that the Moran test based on asymptotic theory may not be reliable for the SAR model with heteroscedastic disturbances.

When compared with the asymptotic test, the dotted curve representing the spatial bootstrap test is closer to the zero line as the number of bootstrap replications increases. In fact, the size or size distortion of the spatial bootstrap test stabilizes as the number of bootstrap replications increases. The interesting exception is the one with Anselin's Wmat spatial weights matrix as shown in the first row of graphs in Figure 1, where we find a small positive size distortion for all four values of spatial lag parameters. However, the extent of over-rejection is minimal compared to the 5% nominal level of significance. In general, when the number of bootstrap replications increases over 399, the size or size distortion of the spatial bootstrap test stabilizes. In order to effectively test for spatial dependence by the asymmetric wild bootstrap method, we find that 399 bootstrap replications are enough, and a larger number is not necessary.

Subsequently, for SAR models with mixture disturbances, the size distortion of both the spatial bootstrap and asymptotic test are plotted in the grids of 3 by 4 of Figure 2. The meanings of the curves in Figure 2 are the same as in Figure 1. The corresponding details of the simulation results are given in the Appendix Tables A.4-A.6.

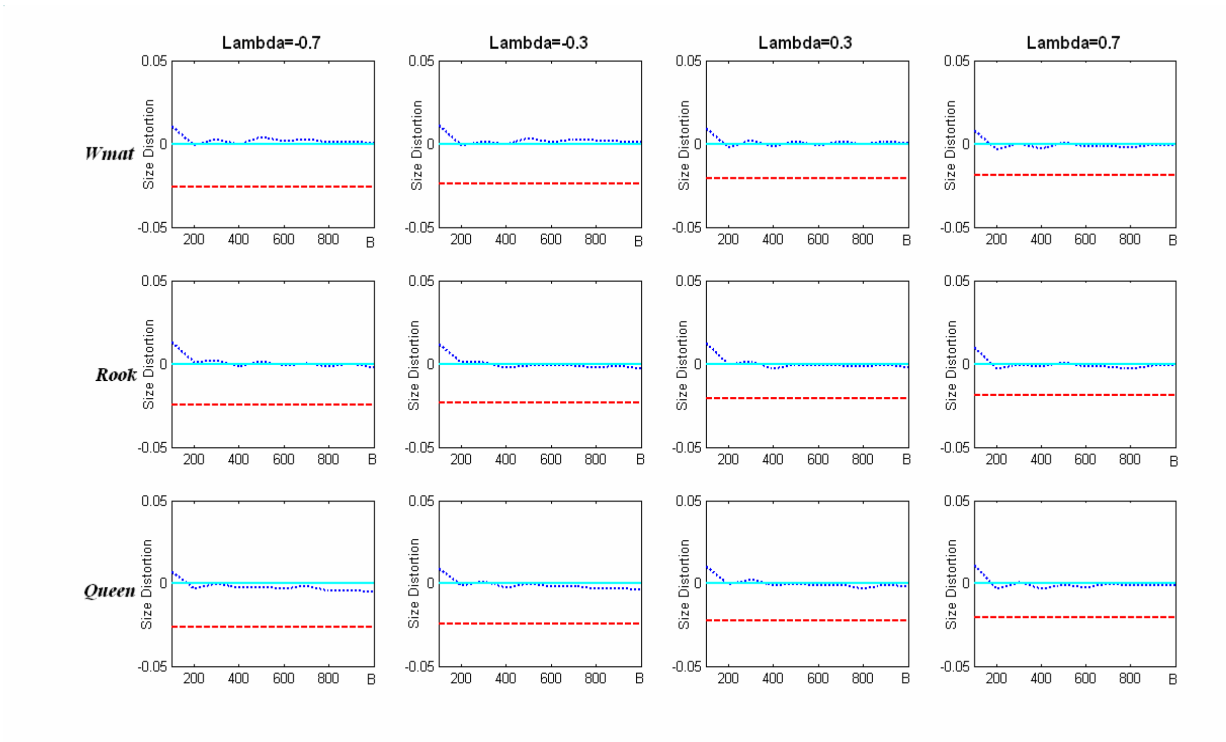

Figure 2. Size Distortion of the Spatial Bootstrap and Asymptotic Tests: SAR Model with Mixture Disturbances, N=49, M=5000

Reading from Figure 2, the graphical patterns of size distortions for models with unknown mixture disturbances are essentially the same as those with heteroscedastic disturbances. In almost all cases, the size distortion of the spatial bootstrap test is very close to zero. Moreover, using Anselin's Wmat spatial weights matrix, we do not find any over-rejection as we have seen in the models with heteroscedastic disturbances. In particular, when the number of bootstrap replications reaches 399 and beyond, the size distortion of the spatial bootstrap test is essentially nil. On the other hand, the asymptotic test, which under-rejects in all cases, is less reliable than the spatial bootstrap test based on the size distortion.

In summary, for heteroscedastic and non-normal models, the size distortion (in absolute value) of the asymptotic Moran test is larger than that of the spatial bootstrap test. The performance of the spatial bootstrap test improves as the number of bootstrap replications increases. In general, the spatial bootstrap test performs well with 399 bootstrap replications. Based on Moran's I statistic, the spatial bootstrap test is an effective alternative to the theoretical asymptotic test.

4.3.2 Power of the Spatial Bootstrap Test

In this section, we investigate the power of the spatial bootstrap and asymptotic tests for different model assumptions, spatial contiguity configurations, and sample sizes. We have concluded in the previous section that the size of bootstrap tests stabilizes when there are 399 or more bootstrap replications. Therefore, the following power results are based on 399 bootstrap replications using asymmetric wild bootstrap methods for two non-normal model structures. We note that for power evaluation, the DGP is assumed to be the alternative SARAR model under consideration as the null SAR model is rejected[7]. The power curves trace the powers of the spatial bootstrap and asymptotic tests over the spatial autocorrelation parameters ranging from -0.9 to 0.9 with 0.1 in each interval.

4.3.2.1 Heteroscedastic Disturbances

For models with heteroscedastic disturbances, the power of the spatial bootstrap test and asymptotic test are compared for four selected spatial lag parameters (-0.7, -0.3, 0.3, 0.7) and three different configurations of a spatial weights matrix. Spatial contiguity structures include a Wmat matrix, Rook matrix and Queen matrix for N=49. In the 3 by 4 grid of graphs in Figure 3, the dotted and solid curves denote the power curves for the spatial bootstrap test and asymptotic test, respectively. The horizontal axis of each graph is the value of spatial autocorrelation parameters (ρ or Rho = -0.9, -0.8, …, 0.8, 0.9). The vertical axis represents the power value of the tests. The higher and steeper the curve is, the larger the power is. We find in all cases that the dotted curves are higher and steeper than the corresponding solid curves. This indicates that the power of the spatial bootstrap test is higher than that of the asymptotic test, independent of spatial lag and autocorrelation parameters for all three different spatial contiguity configurations under consideration. The detailed numerical results are listed in Appendix Tables A.7-A.9.

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[7] We have run the simulation according to SARMA alternatives, and have obtained similar results and conclusions. Due to space limitations, we report only the results of the SARAR models.

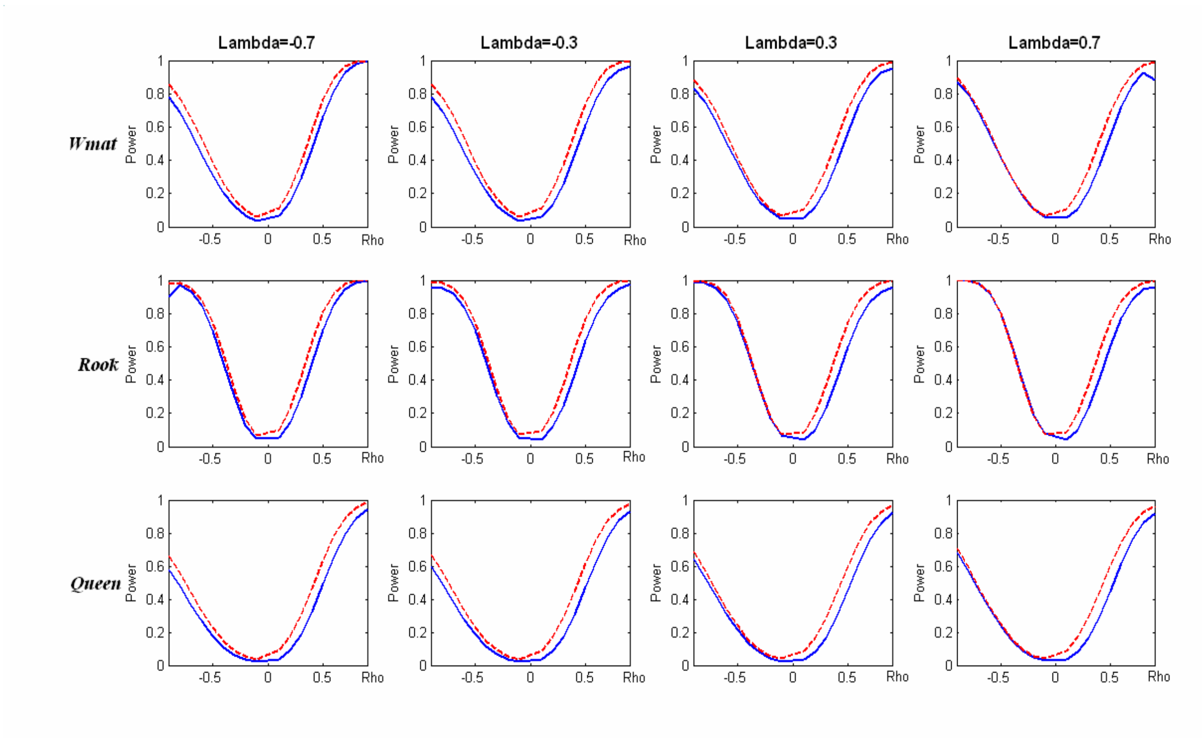

Figure 3. Power of the Spatial Bootstrap and Asymptotic Tests: SARAR Alternative Model with Heteroscedastic Disturbances, N=49, B=399, M=5000

Next, we consider the effects of sample size (N = 36, 81, and 121) and two regular spatial weights matrices (Rook and Queen) on the power of the spatial bootstrap and asymptotic tests. Figure 4 portrays the power curves when the DGP is obtained from a SARAR model with a Rook spatial contiguity matrix. Similarly, Figure 5 displays the power curves for the SARAR model with a Queen spatial contiguity matrix. It is observed that the power of the spatial bootstrap test is equivalent to or slightly higher than that of the asymptotic test in all cases for both Rook and Queen spatial contiguities. When the spatial weights matrix is defined by the Rook contiguity, the power curves are in V form as shown in Figure 4, while U-shaped curves are observed for the cases with the Queen spatial contiguity in Figure 5. For the models with a smaller sample size (N=36), as shown in the first row of graphs in Figure 4 and Figure 5, the spatial bootstrap test outperforms the asymptotic test independent of the values of spatial lag parameters, positive or negative. When the sample size increases, the power curves become steeper and more symmetric. The numerical results are given in Appendix Tables A.13-A.15.

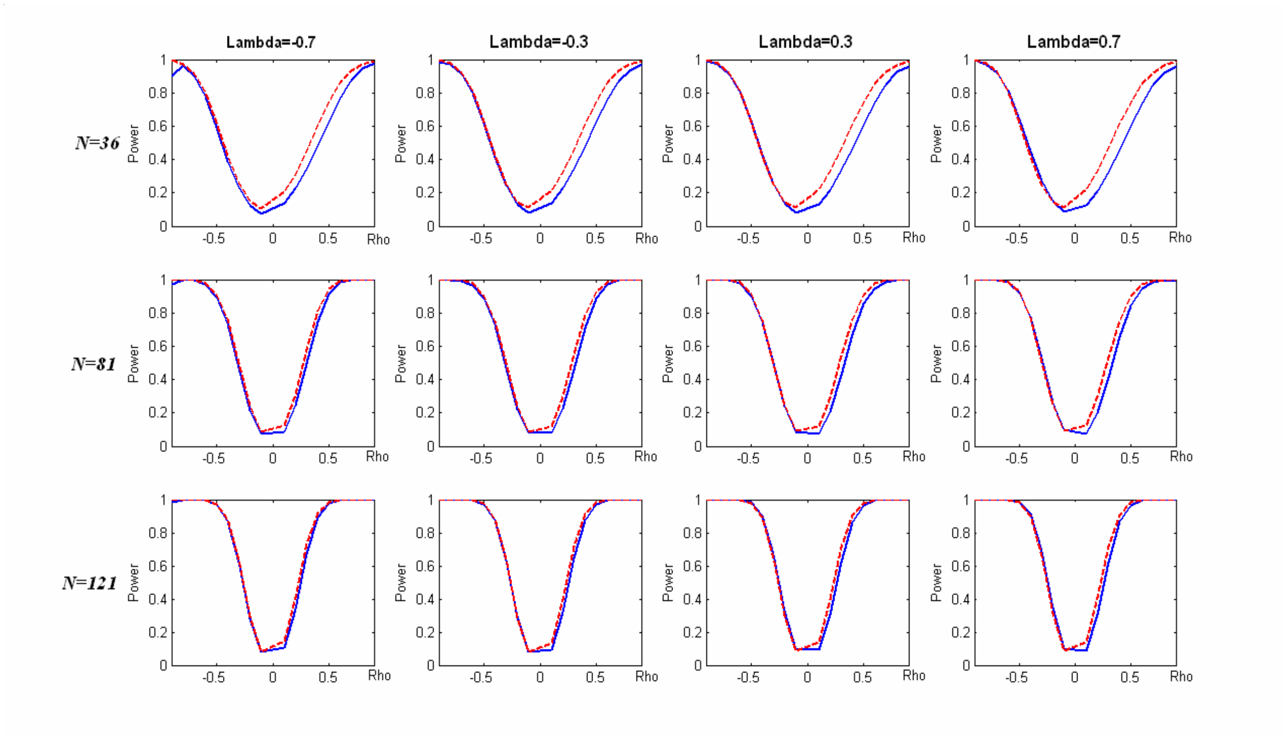

Figure 4. Power of the Spatial Bootstrap and Asymptotic Tests: SARAR Model with Heteroscedastic Disturbances, W=Rook, B=399, M=5000

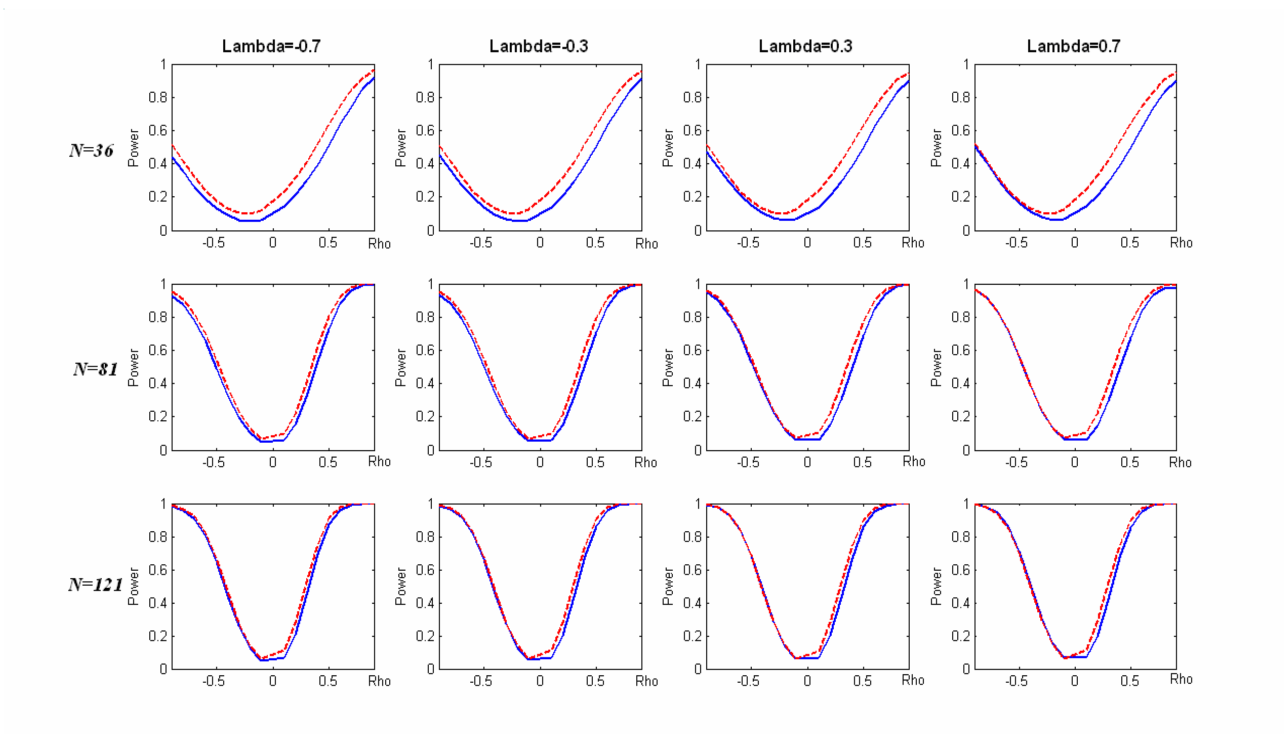

Figure 5. Power of the Spatial Bootstrap and Asymptotic Tests: SARAR Model with Heteroscedastic Disturbances, W=Queen, B=399, M=5000

4.3.2.2 Non-Normal Mixture Disturbances

Next, we consider the alternative SARAR model with mixture disturbances. The conclusion in favor of spatial bootstrap tests is similar to the cases of heteroscedastic disturbances. We study and compare the power of spatial bootstrap and asymptotic tests from the two aspects.

First, the effects of three spatial contiguity configurations and four spatial lag parameters are investigated for N=49. Figure 6 shows the spatial bootstrap test to be an effective alternative to the asymptotic test based on Moran's I statistic at 5% nominal level of significance. The U-shaped power curves for the spatial bootstrap test are similar to those of the heteroscedastic disturbances. In particular, the spatial bootstrap test performs better in the positive range of the spatial autocorrelation parameters. The corresponding numerical results are given in the Appendix Tables A.16-A.18.

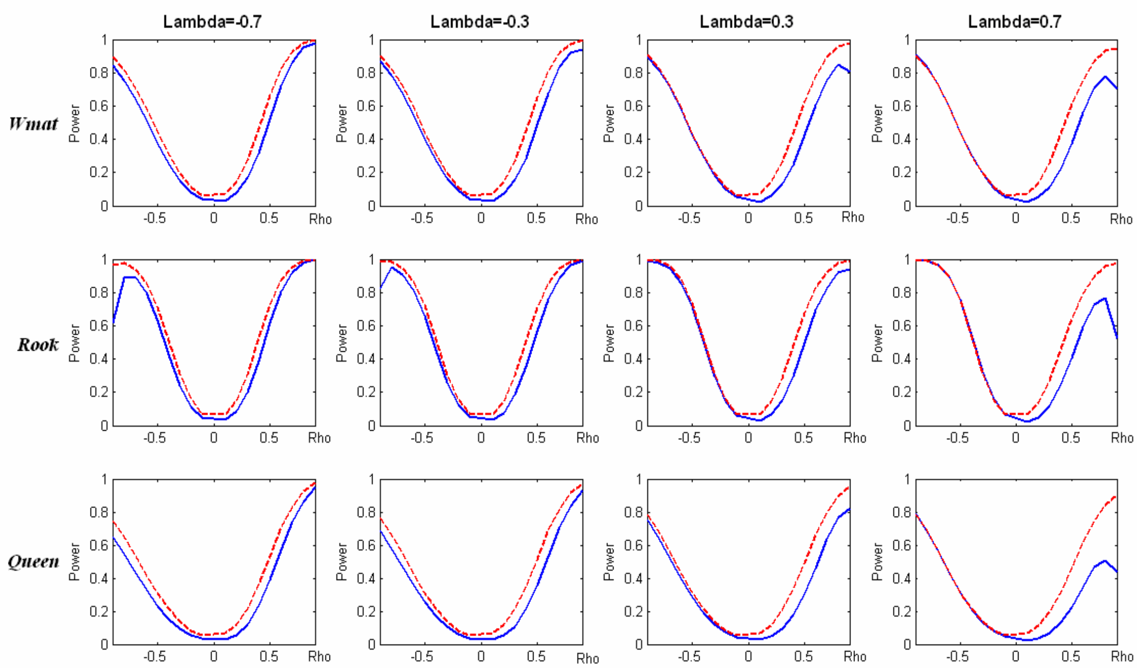

Figure 6. Power of the Spatial Bootstrap and Asymptotic Tests: SARAR Alternative Model with Non-Normal Mixture Disturbances, N=49, B=399, M=5000

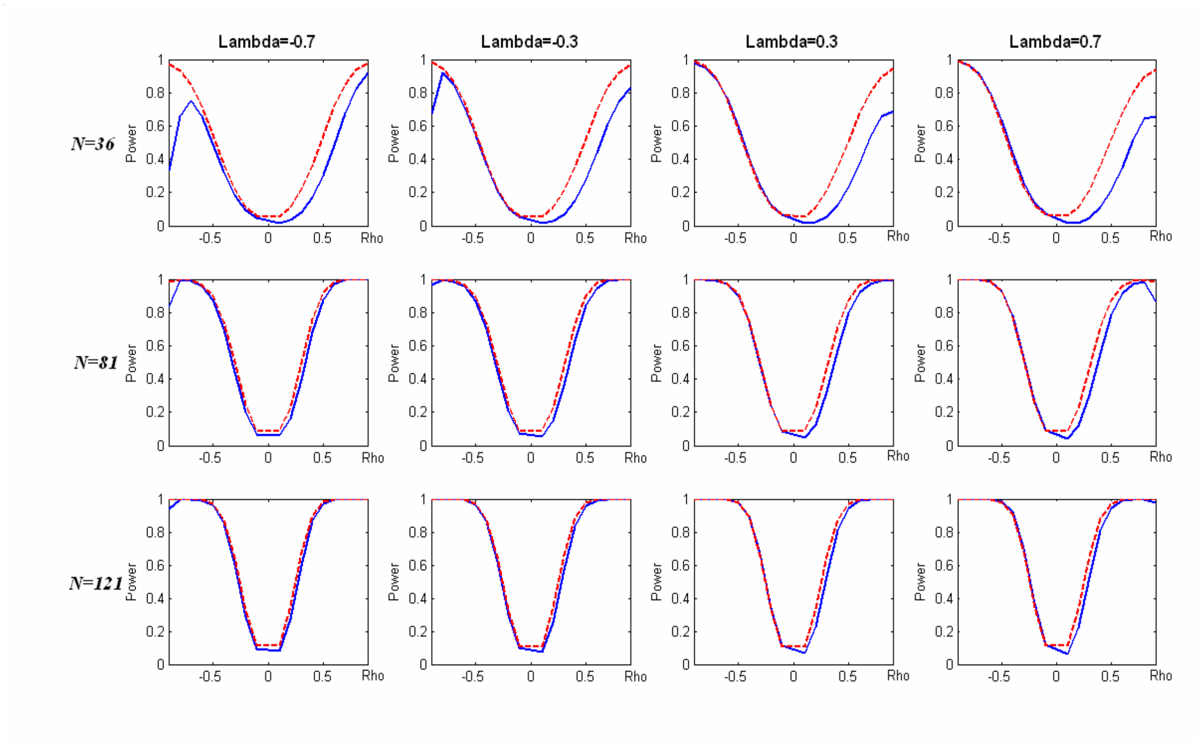

Figure 7. Power of the Spatial Bootstrap and Asymptotic Tests: SARAR Alternative Model with Non-Normal Mixture Disturbances, W=Rook, B=399, M=5000

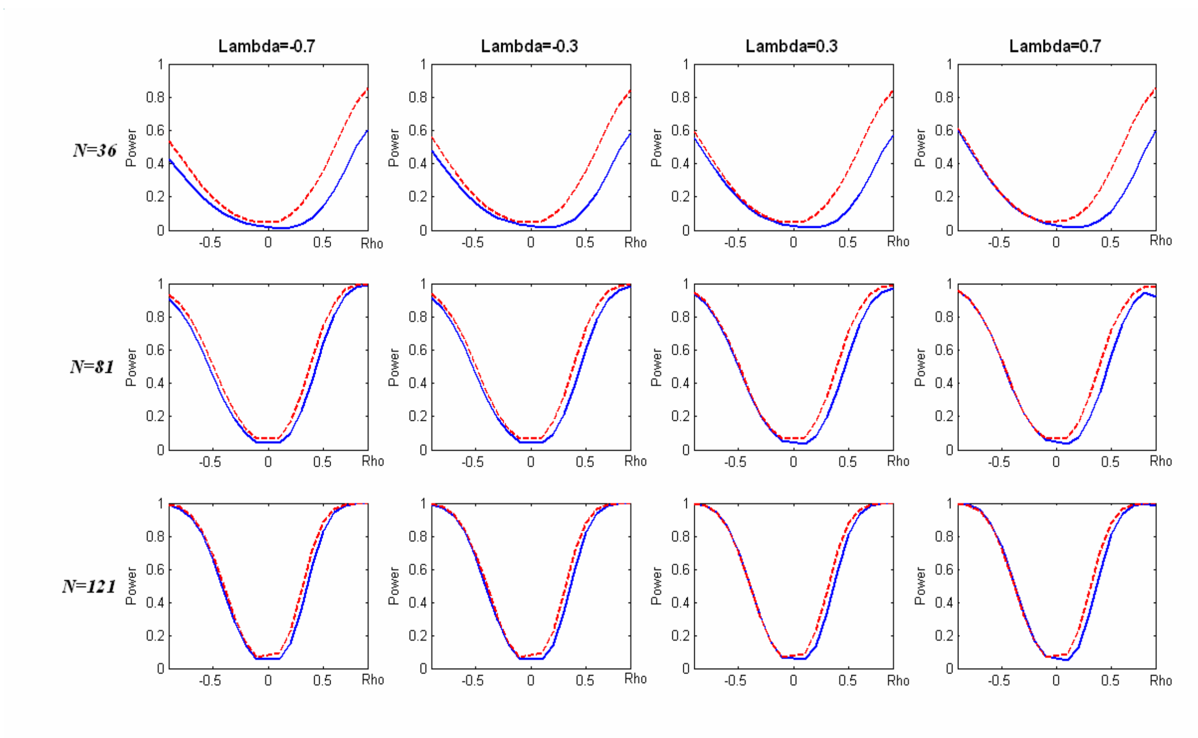

Figure 8. Power of the Spatial Bootstrap and Asymptotic Tests: SARAR Alternative Model with Non-Normal Mixture Disturbances, W=Queen, B=399, M=5000

In addition to the case of N=49, the power of the spatial bootstrap and asymptotic tests is compared for different sample sizes (N = 36, 81, 121) and two regular spatial contiguity matrices (Rook and Queen). The corresponding graphs are presented in Figure 7 and Figure 8. Tables of numerical results are shown in the Appendix Tables A.19-A.24.

We find that the spatial bootstrap test performs well for the alternative SARAR models with non-normal mixture disturbances. For cases using the Rook spatial contiguity matrix, Figure 7 shows that the power curves in V-shape are similar to those of the heteroscedastic models in Figure 4. Similarly, cases using a Queen configuration of the spatial contiguity matrix show power curves in a U-shape (Figure 8), as in heteroscedastic models of Figure 5. In all cases of the alternative SARAR model with non-normal mixture disturbances, the powers of the spatial bootstrap tests are higher than those of the asymptotic tests, particularly with a small sample in the positive range of spatial autocorrelation parameters, shown in the first row of graphs in Figure 7 and Figure 8.

In summary, based on our extensive Monte Carlo experiments, the power of the spatial bootstrap test is higher than that of the asymptotic test for non-normal models with heteroscedastic or mixture disturbances. The advantage of the spatial bootstrap test is clear for applications with a small sample and dense spatial contiguity.

5. Conclusion

In this paper, we apply asymmetric wild bootstrap methods to test spatial correlation in a Cliff-Ord type spatial lag model with spatial autoregressive disturbances. It is known that the asymptotic test performs poorly with a large size distortion and weak power when the classical normality assumption of the model is not satisfied. Based on Moran's I test statistic computed from the 2SLS regression residuals of the SAR model, we demonstrate the use of bootstrap methods for diagnostic testing of the model specification. Our Monte Carlo experiments indicate that for more realistic applications with heteroscedastic and non-normal innovations, the residual-based spatial bootstrap test works better than the asymptotic test, particularly in cases with a small size and dense spatial contiguity.

We compare the size distortion of spatial bootstrap and asymptotic tests, based on Moran's I statistic, for various structures of spatial contiguity and spatial lag parameters. For the non-normal models with heteroscedastic and mixture disturbances, the size distortion of the asymptotic test is far below the ideal zero value. However, a mere 399 bootstrap replications stabilizes the size of bootstrap tests to the ideal level.

Considering the alternative model with spatial autoregressive disturbances, we study the power of the spatial bootstrap and asymptotic tests. We find that the choice of model specification and spatial contiguity structure has an impact on the power of the bootstrap and asymptotic tests, especially for cases with a smaller sample. For spatial models with the regular lattice Queen contiguity or the more practical irregular lattice Wmat matrix, the power of the spatial bootstrap test is generally higher than that of the asymptotic test, mostly occurring in the positive range of the spatial autoregressive parameters. The U-shaped power curves of the spatial bootstrap and asymptotic tests are steeper and more symmetric as the sample increases.

We conclude from our Monte Carlo experiments that the spatial bootstrap test is superior to the asymptotic test for general non-normal models, resulting in a smaller size distortion and larger power. In fact, using bootstrap methods for diagnostic testing the spatial dependence in a SAR model framework is effective and simple to implement. For future research, bootstrap methods will be applied to the more general LM-based tests for diagnostic checking and validation of a spatial econometric model.

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Appendix

Tables of simulations results A.1-A.24 are available upon request.